Stability and Uniqueness for Inverse Problems for Partial Differential Equations by Carleman Estimates

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Foreword

- Theoretical studies for inverse problems
- Numerical methods for inverse problems

IP numerics without theory $\Longrightarrow$ No quality assurance
IP theory without numerics $\Longrightarrow$ Mathematical toys
Hopefully my theoretical achievements may be useful also for numerical approaches.

Main technics: Carleman estimate
Formulation: Inverse problems for PDE with single measurement

What is Carleman estimate?
How to use Carleman estimate for IP?

Contents:

- Part I. Main methodology
- Part II. Recent results by me with Prof. Imanuvilov
  - inverse problems for parabolic equations
  - inverse problems for hyperbolic equations
  - unique continuation for Schödinger equation

Part I. Description of main methodology: case of first-order equations

Part I is from a joint work with

Professor Piermarco Cannarsa
(Università degli Studi di Roma "Tor Vergata")
Professor Giuseppe Floridia
(Sapienza Università di Roma)

§1. Introduction

Main target: Transport equation:

$$\partial_t u(x,t) + (H(x)\cdot\nabla u) + p(x,t)u = 0 \quad \text{in } Q := \Omega\times(0,T)$$

$\Omega \subset \mathbb{R}^d$: bounded smooth domain,
$|H| \neq 0$ on $\overline{\Omega}$, $H \in C^1(\overline{\Omega})$, $p \in L^\infty$
Let $\Gamma \subset \partial\Omega$ be given subboundary.

Observability: Determine $u(\cdot,0)$ in Ω by $u|_{\Gamma\times(0,T)}$.

Inverse coefficient problem:
$u|_{\Gamma\times(0,T)}$ and $u(\cdot,0)|_\Omega \quad\Longrightarrow p(x)$ for $x\in\Omega$
in $\partial_t u(x,t) + H(x)\cdot\nabla u + p(x)u = 0$.

References:

[1] P. Cannarsa, G. Floridia, F. Gölgeleyen, M. Yamamoto, Inverse coefficient problems for a transport equation by local Carleman estimate, Inverse Problems 35 (2019).

[2] O. Imanuvilov and M. Yamamoto, Inverse problems for a compressible fluid system (2020).

[3] F. Gölgeleyen and M. Yamamoto, Stability for some inverse problems for transport equations, SIAM J. Math. Anal. 48 (2016).

[4] P. Gaitan and H. Ouzzane, Inverse problem for a free transport equation using Carleman estimates, Appl. Anal. 93 (2014).

Basic setting

Main equation

$$\partial_t u(x,t) + (H(x) \cdot \nabla u) + p(x,t)u = 0 \quad \text{in } Q := \Omega \times (0,T)$$

Standing notations

- $\Omega \subset \mathbb{R}^d$: smooth bounded domain
- $Q := \Omega \times (0,T)$.
- $S^{d-1} := \{x \in \mathbb{R}^d;\ |x| = 1\}$
- $\nu = \nu(x)$: unit outward normal vector to $\partial\Omega$ at x.

§2. Carleman estimate: Non-rotating case

Assumption: $\exists v \in S^{d-1} := \{x \in R^d;\ |x| = 1\}$ such that $(H(x) \cdot v) > \exists\delta_0 > 0$ in Q

$\Longrightarrow \{\frac{H(x)}{|H(x)|};\ x \in \Omega\} \subset \frac{S^{d-1}}{2}$.

Set $\varphi(x,t) := |x + rv|^2 - \beta t,\ r >> 1,\ \beta << 1,$ $\Sigma_+ := \{(x,t) \in \partial\Omega \times (0,T);\ (H(x) \cdot \nu(x)) > 0\}$.

Theorem 1 (Carleman estimate) $\exists s_0 > 0$ *and* $\exists C > 0$ *such that*

$$s\int_\Omega |u(x,0)|^2 e^{2s\varphi(x,0)}dx + s^2\int_Q |u|^2 e^{2s\varphi(x,t)}dxdt$$

$$\le C\int_Q |(\partial_t + (H\cdot\nabla))u|^2 e^{2s\varphi(x,t)}dxdt + Cs\int_{\Sigma_+} |u|^2 e^{2s\varphi(x,t)}dSdt$$

$$+ Cs\int_\Omega |u(x,T)|^2 e^{2s\varphi(x,T)}dx \quad \textit{for all } s \ge s_0.$$

Important characters of Carleman estimates

- Uniform L^2-weighted estimates in large parameter $s > 0$.
- Lower-order terms of PDE do not affect.

Proof. $(H \cdot v) > 0$ on $\overline{\Omega} \Longrightarrow$
$B(x,t) := (\nabla\varphi \cdot H) - \beta = 2(x + rv \cdot H) - \beta \geq O(r) - \beta > 0$ on $\overline{\Omega}$
for $r >> 1$ and $\beta << 1$. Set $w := ue^{s\varphi} \Longrightarrow$

$$Pw := e^{s\varphi}(\partial_t + (H \cdot \nabla))(e^{-s\varphi} w) = (\partial_t w + (H \cdot \nabla w)) - sBw.$$

Then, neglecting $|\partial_t w + (H \cdot \nabla w)|^2$, we have

$$\int_Q |\partial_t u + (H \cdot \nabla u)|^2 e^{2s\varphi} dxdt = \int_Q |Pw|^2 dxdt$$
$$\geq s^2 \int_Q B^2 w^2 dxdt - 2s \int_Q Bw(x,t)(\partial_t w + (H \cdot \nabla w))dxdt$$

Use integration by parts and $B > 0$ on $\overline{\Omega}$ $\blacksquare$

§3. Methodology from Carleman estimate to inverse problems

Method: originally by Bukhgeim-Klibanov (1981). We rely on
Simplification by Huang, Imanuvilov and Yamamoto (2020)

Assume that $\exists v \in S^{d-1}$ such that $(H \cdot v) \ge \exists \delta_0 > 0$ on $\overline{\Omega}$ for
$\partial_t u(x,t) + (H(x) \cdot \nabla u) + p(x)u(x,t) = 0$.

Theorem 2 (observability). *Let* $T > \frac{1}{\delta} \sup_{x\in\Omega} |x + rv|^2$. *Then*
$\|u(\cdot,0)\|_{L^2(\Omega)} \le C\|u\|_{L^2(\partial\Omega\times(0,T))}$.

Here $\delta > 0$ depends on δ_0 and Ω.

Remark. Hölder stability: Let $\|u(\cdot,T)\|_{L^2(\Omega)} \le M_0$. Then $\exists C > 0$ and $\exists \kappa \in (0,1)$ such that

$$\|u(\cdot,0)\|_{L^2(\Omega)} \le C(\|u\|^{\kappa}_{L^2(\Sigma_+)} + \|u\|_{L^2(\Sigma_+)}).$$

Σ_+: outgoing subboundary of $\partial\Omega \times (0,T)$.

Inverse coefficient problem

Let $u = u(p) \in H^1(Q)$ satisfy

$$\begin{cases} \partial_t u + (H(x)\cdot \nabla u(x,t)) + p(x)u = 0, \\ u(x,0) = u_0(x), \qquad x \in \Omega,\ 0 < t < T. \end{cases}$$

We assume $\|p\|_{L^\infty(\Omega)}, \|q\|_{L^\infty(\Omega)} \le M$: arbitrarily fixed constant.

Theorem 3.

Let

$$T > \frac{1}{\delta} \sup_{x\in\Omega} |x + rv|^2, \quad |u_0| > 0 \quad \textit{on } \overline{\Omega}.$$

Then

$$\|p - q\|_{L^2(\Omega)} \le C\|\partial_t u(p) - \partial_t u(q)\|_{L^2(\partial\Omega\times(0,T))}$$

under a priori boundness on $u(p), u(q)$ with suitable norms.

Proof of the observability inequality (Theorem 2)

$\partial_t u + ((H(x)\cdot\nabla u) + p(x,t)u = 0$ in Q.

Lemma 1 (energy estimate). $\|u(\cdot,T)\|_{L^2(\Omega)} \le C(\|u(\cdot,0)\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega\times(0,T))})$.

Carleman estimate $\Longrightarrow$

$$\int_\Omega |u(x,0)|^2 e^{2s\varphi(x,0)}\,dx \le C\int_{\partial\Omega\times(0,T)} |u|^2 e^{2s\varphi}\,dxdt + C\int_\Omega |u(\cdot,T)|^2 e^{2s\varphi(x,T)}\,dx$$

Then $T > \frac{1}{\delta}\sup_{x\in\Omega}|x+rv|^2 \Longrightarrow$

- $-\mu_0 := \sup_{x\in\Omega}\varphi(x,T) = \sup_{x\in\Omega}|x+rv|^2 - \beta T < 0$ if we choose small $\beta \sim \delta$.
- $\varphi(x,0) \ge 0$

$\Longrightarrow$

$$\int_\Omega |u(x,0)|^2\,dx \le Ce^{Cs}\|u\|^2_{L^2(\partial\Omega\times(0,T)} + Ce^{-2s\mu_0}\int_\Omega |u(x,T)|^2\,dx$$

Lemma 1 $\Longrightarrow$

$$\|u(x,0)\|^2_{L^2(\Omega)} \le Ce^{Cs}\|u\|^2_{L^2(\partial\Omega\times(0,T))}$$
$$+Ce^{-2s\mu_0}\|u(\cdot,0)\|^2_{L^2(\Omega)} + Ce^{-2s\mu_0}\|u\|^2_{L^2(\partial\Omega\times(0,T))}$$

Take $s > 0$ large $\Longrightarrow$ Absorb the second term on right side into left side ■

Part II. Recent works on inverse problems by Carleman estimates

With Professor Oleg Imanuvilov
(Colorado State University)

§1. Inverse parabolic problems

Let $\Omega \subset \mathbb{R}^d$: bounded domain, ν: unit outward normal vector to $\partial\Omega$

$$\begin{cases} \partial_t u = \mathrm{div}\,(a(x)\nabla u) + c(x)u(x,t), \quad (x,t) \in \Omega \times (0,T), \\ \partial_\nu u|_{\partial\Omega} = 0, \\ u(\cdot, t_0) = u_0 \quad \text{in } \Omega \end{cases}$$

Here $a > 0$ on $\overline{\Omega}$, $\in C^3(\overline{\Omega})$. Let $\Gamma \subset \partial\Omega$ be a subboundary.
Inverse problem: $u|_{\Gamma\times(0,T)} \Longrightarrow c(x)$

References (not comprehensive)
Case I: $0 < t_0 < T$.
Bukhgeim-Klibanov (1981): uniqueness
Imanuvilov-Yamamoto (1998): Lipschitz stability in Ω (global)
My life has been with inverse parabolic problems!
Yamamoto-Zou (2001): stability and numerical methods

Case II: $t_0 = T$. no data after T.
Imanuvilov and Yamamoto (2023): global Lipschitz stability

Case III: $t_0 = 0$ (IP for initial boundary value problem)

- Klibanov (1992): uniqueness if $\Gamma \supsetneqq$ "half of $\partial\Omega$" and $\left|\frac{(\nabla a \cdot (x - x_0))}{a(x)}\right| < 2$ with some $x_0 \notin \overline{\Omega}$.
- Suzuki-Murayama (1980): uniqueness if any eigenmode of $u_0 > 0$ is not zero for one dimensional case by Gel'fand-Levitan theory (inverse eigenvalue problem)
- Imanuvilov and Yamamoto (2023): uniqueness if $u_0 \neq 0$ is "very smooth".

Case II: $t_0 = T$.

$$\begin{cases} \partial_t u = \text{div}\,(a\nabla u) + c(x)u, & x \in \Omega,\ 0 < t < T, \\ \partial_\nu u|_{\partial\Omega} = 0, & u(\cdot, T) = u_0. \end{cases}$$

Let $u(c,b)(x,t)$ be solution to

$$\begin{cases} \partial_t u = \text{div}\,(a\nabla u) + c(x)u, \\ \partial_\nu u|_{\partial\Omega} = 0, \quad u(\cdot, 0) = b. \end{cases}$$

Let $H^\gamma(\Omega)$ be Sobolev-Slobodecki space, $\mathcal{P} := \{c \in C^\gamma(\overline{\Omega});\ \|c\|_{C^\gamma(\overline{\Omega})} \le M\}$,

$\mathcal{B} := \{b \in C^{2+\gamma}(\overline{\Omega});\ \partial_\nu b|_{\partial\Omega} = 0,\ \|b\|_{C^{2+\gamma}(\overline{\Omega})} \le M,\ b \ge \delta_0\}$ with arbitrarily chosen $M > 0$, $\delta_0 > 0$ and $0 < \gamma < 1$.

Theorem 1.1 (Imanuvilov-Y. 2023). *Let $\Gamma \subset \partial\Omega$ be arbitrarily chosen and $0 < \theta < \gamma$. Then*

$$\|c_1 - c_2\|_{H^\theta(\Omega)} \le C(\|u(c_1,b_1) - u(c_2,b_2)\|_{H^1(\Gamma\times(0,T))} + \|(u(c_1,b_1) - u(c_2,b_2))(\cdot,T)\|_{H^{2+\theta}(\Omega)})$$

for all $c_1, c_2 \in \mathcal{P}$ and $b_1, b_2 \in \mathcal{B}$.

Key for Proof.

Imanuvilov-Yamamoto (2023): ArXiv 2211.11930

- Global Carleman estimate with the weight:

$$\frac{e^{\lambda d(x)} - e^{2\lambda \|d\|_{C(\overline{\Omega})}}}{t(T-t)}.$$

- compactness-uniqueness argument

Case III: $t_0 = 0$. Not completely solved.

$$u(c): \begin{cases} \partial_t u = \text{div}\,(a\nabla u) + c(x)u, \\ \partial_\nu u|_{\partial\Omega} = 0, \quad u(\cdot, 0) = u_0. \end{cases}$$

with fixed u_0

Inverse problem. *Determine $c(x)$, $x \in \Omega$ by $u|_{\Gamma\times(0,T)}$.*

We have no adequate Carleman estimates in this case.

Let $-Av := \mathrm{div}\,(a(x)\nabla v) + c(x)v$ with domain $\{v \in H^2(\Omega);\ \partial_\nu v|_{\partial\Omega} = 0\}$.

Theorem 1.2 (Imanuvilov-Y. 2023). *Let*

$$u_0 \in \mathcal{R}(\exp(-A^{\frac{2}{3}+\varepsilon}\tau)), \quad u_0(x) \ne 0 \quad \textit{for all } x \in \overline{\Omega}$$

with some $\varepsilon > 0, \tau > 0$ and let $\Gamma \subset \partial\Omega$ be arbitrarily chosen subboundary. Then $u(c_1) = u(c_2)$ on $\Gamma \times (0,T)$ implies $c_1 = c_2$ in Ω.

Remarks.

(1) $u_0 \in \mathcal{R}(\exp(-A^{\frac{2}{3}+\varepsilon}\tau)) \Longleftrightarrow \sum_{k=1}^\infty (u_0, \varphi_k)^2 \exp(2\lambda_k^{\frac{2}{3}+\varepsilon}\tau) < \infty$

$\Longrightarrow$ very strong regularity.

Here $A\varphi_k = \lambda_k\varphi_k$ and $\|\varphi_k\|_{L^2(\Omega)} = 1$ for $k \in \mathbb{N}$

(2) If $u_0 \in \mathcal{R}(e^{-A\tau})$, the uniqueness is trivial: $u(\cdot,t)$ can be analytically extended to $(-\tau, 0)$, and is reduced to Case I: $0 < t_0 < T$.

Technical remarks.

1. Bukhgeim-Klibanov method by Carleman estimate does not work for $t_0 = 0$.

2, Transfer to other type of equation:

(A) Reznitskaya transform (similar to Laplace transform)

$\Longrightarrow$ reduction of inverse parabolic problem to inverse hyperbolic problem

However, inverse hyperbolic problem by Carleman estimate requires

- geometric constraints on observation subboundary Γ
- extra conditions on principal term $a(x)$.

$\Longrightarrow$ not natural condition for inverse parabolic problem.

(B) Some integral transform reducing inverse parabolic problem to inverse elliptic problem is known (e.g., Romanov: ”Inverse Problems of Mathematical Physics” (1987)). However not applicable to our case.

§2. Inverse hyperbolic problem (I)

$$u(c):\begin{cases} \partial_t^2 u = \mathrm{div}\,(a\nabla u) + c(x)u, \\ \partial_\nu u|_{\partial\Omega} = 0, \\ u(\cdot,0) = u_0,\ \partial_t u(\cdot,0) = u_1. \end{cases}$$

Inverse hyperbolic problem (global version).

Let $\omega \subset \Omega$: subdomain. Determine c in Ω by $u|_{\omega\times(0,T)}$.

Wave propagation $\Longrightarrow$ $\omega \subset \Omega$ should be large, $T > 0$ should be large.

Local uniqueness: Bukhgeim-Klibanov

Let $\nu(x)$ be outward normal vector to $\partial\Omega$ at x, set
$\mathcal{U}_M := \{c \in W^{1,\infty}(\Omega);\ \|c\|_{W^{1,\infty}(\Omega)} \le M\}$ with arbitrarily fixed $M > 0$.
Theorem 2.1 (Imanuvilov-Y.:2001) *Let*

$$\partial\omega \supset \{x \in \partial\Omega;\ ((x - x_0) \cdot \nu(x)) \ge 0\}$$

$$T > \sup_{\Omega} |x - x_0|$$

for some $x_0 \notin \overline{\Omega}$ *and* $\partial\Omega \setminus \partial\omega$ *is locally convex. Let* $|u_0(x)| > 0$ *for* $x \in \overline{\Omega}$, $\partial_\nu u(c_k)|_{\partial\Omega} = 0$, $k = 1, 2$. *Then* $\|c_1 - c_2\|_{L^2(\Omega)} \sim \sum_{k=1}^2 \|\partial_t^k (u(c_1) - u(c_2))\|_{L^2(\omega\times(0,T))}$
for all $c_1, c_2 \in \mathcal{U}_M$.

Key of Proof. Energy estimate + Carleman estimate:

Set $\varphi(x,t) = e^{\lambda(|x-x_0|^2-\beta t^2)}$ with large $\lambda > 0$. Then $\exists s_0 > 0$ and $\exists C > 0$ such that

$$\int_{-T}^{T}\int_{\Omega}(s|\nabla_{x,t}y|^2 + s^3|y|^2)e^{2s\varphi}\,dxdt$$

$$\le C\int_{-T}^{T}\int_{\Omega}|(\partial_t^2 - \Delta)y|^2 e^{2s\varphi}\,dxdt + C\int_{-T}^{T}\int_{\omega}(s|\partial_t y|^2 + s^3|y|^2)e^{2s\varphi}\,dSdt$$

for all $s > s_0$ and $y \in H^2(\Omega \times (-T,T))$ satisfying $\partial_\nu y|_{\partial\Omega\times(-T,T)} = 0$ and $\partial_t^j y(\cdot, \pm T) = 0$ in Ω, $j = 0, 1$.

Proof is elementary by integration by parts.
Ref: Yamamoto (2023): Rome Lecture Note, Bellassoued and Yamamoto (2017)

§3. Inverse hyperbolic problem (II):variable wave speeds

Consider $r(x)\partial_t^2 u(x,t) = \Delta u$:
Unique continuation: *Let* $\Gamma \subset \partial\Omega$. $u, \nabla u$ *on* $\Gamma \times (0,T)$, $\Longrightarrow$ u *in a subdomain?*

Variable $r(x) \Longrightarrow$ wave speed changes in $x \Longrightarrow$ refraction or cloaking (non-uniqueness)

We need conditions on $r(x)$ and on Γ, T.

Remark.
Carleman estimate $\Longrightarrow$ uniqueness.
So non-uniqueness implies no Carleman estimate.

Carleman estimate for $r(x)\partial_t^2 u - \Delta u = F$. Same recipe of the proof as for the transport equation.

Let $Q_\pm := \Omega \times (-T, T)$ and $r > 0$ on $\overline{Q_\pm}$, $\in C^2(\overline{Q_\pm})$.

Set $\varphi(x,t) = d(x) - \beta t^2$ with $d \in C^2(\overline{\Omega})$ and $0 < \beta << 1$: chosen later.

First Step. Set $w := ue^{s\varphi}$ and $Pw := e^{s\varphi}(r(x)\partial_t^2 - \Delta)(e^{-s\varphi}w) \Longrightarrow$

Installed system with weight:

$$Pw = e^{s\varphi}F, \quad \int_{Q_\pm} |u|^2 e^{2s\varphi} dxdt = \int_{Q_\pm} |w|^2 dxdt, \quad \text{etc.,}$$

$\Longrightarrow$ lower estimate of $\|Pw\|^2_{L^2(Q_\pm)} = \int_{Q_\pm} |(r\partial_t^2 - \Delta)u|^2 e^{2s\varphi} dxdt$.

Decompose $P = P_1 + P_2$ suitably.

Second Step. $||Pw||^2_{L^2(Q_\pm)} = ||P_1 w + P_2 w||^2_{L^2(Q_\pm)} \ge 2(P_1 w,\ P_2 w)_{L^2(Q_\pm)}$.

Then

- integartion by parts
- salvage $s^3||w||^2_{L^2(Q_\pm)} + s||\nabla_{x,t} w||^2_{L^2(Q_\pm)}$
- $s > 0$: large, $\beta > 0$: small $\Longrightarrow s = o(s^3)$, $\beta = o(1)$, etc.$\Longrightarrow$
  Absorb any terms of lower orders in s and larger orders in β.

$\Longrightarrow$ estimate (II).

However we can obtain:

$(P_1 w, P_2 w)_{L^2(Q_\pm)} \ge Cs \int_{Q_\pm} (\nabla r \cdot \nabla\varphi)|\partial_t w|^2 dxdt + \cdots +$ [boundary terms]!

However, $(\nabla r \cdot \nabla\varphi) > 0$ does not hold even for $r \equiv 1 \Longrightarrow$ no estimates of $|\partial_t w|^2$.

Third Step: Auxiliary estimate. (not necessary for first-order equations)

$[Pw = Fe^{s\varphi}] \times (-sw)$ (usual energy estimate for wave equation) $\Longrightarrow$ estimate (III).
(II) $+\theta \times (III) \Longrightarrow$ We can salvage $s^3 \|w\|^2_{L^2(Q_\pm)} + s\|\nabla_{x,t} w\|^2_{L^2(Q_\pm)}$.

For salvage, we need condition on r and choice of d in $\varphi(x,t) = d(x) - \beta t^2$.

Consider $r(x)\partial_t^2 u - \Delta u - B(x,t)\cdot\nabla_{x,t}u - c(x,t)u = F$ in $Q_\pm$ where $r \in C^2(\overline{Q_\pm})$, > 0 and $B, c \in L^\infty(Q_\pm)$. Let $\sigma_0 > 0$ be a constant such that $(\partial_i\partial_j d)_{1\le i,j\le d} \ge \sigma_0$ on $\overline{\Omega}$.

Condition on r.

$$\begin{cases} 0 < \theta < 2, \quad \dfrac{(\nabla r \cdot \nabla d)}{r(x)} > -\theta\sigma_0, \\ (\nabla(|\nabla d|^2)\cdot\nabla d) + \theta\sigma_0|\nabla d|^2 > 0 \quad \text{on } \overline{\Omega}. \end{cases}$$

Set $\varphi(x,t) := d(x) - \beta t^2$ with $0 < \beta << 1$.

Theorem 3.1.

$$\int_{Q_\pm}(s|\nabla_{x,t}u|^2 + s^3|u|^2)e^{2s\varphi}\,dxdt \le C\int_{Q_\pm}|F|^2 e^{2s\varphi} + C\int_{\partial\Omega\times(-T,T)}(s|\partial_{x,t}u|^2 + s^3|u|^2)e^{2s\varphi}\,dSdt$$

$$+C\int_\Omega (s|\nabla_{x,t}u(x,\pm T)|^2 + s^3|u(x,\pm T)|^2)e^{2s\varphi(x,T)}\,dx$$

for all large $s > 0$.

Case 1. $r \equiv 1$: Consider $\partial_t^2 u - \Delta u +$ (l.o.t) $= F$.

Set $\varphi(x,t) = |x - x_0|^2 - \beta t^2$ with $x_0 \notin \overline{\Omega}$ and $0 < \beta < 1$.
Then $\sigma_0 = 2$ and
$\nabla(|\nabla d|^2) \cdot \nabla d) + \theta\sigma_0 |\nabla d|^2 = (16 + 8\theta)|x - x_0|^2 > 0$ by $x_0 \notin \overline{\Omega}$.

Case 2. $\partial_1 r \geq \exists \delta_0 > 0$ on $\overline{\Omega}$.
Set $d(x) = \alpha(x_1 - \ell)^2 + |x'|^2$, where $x := (x_1, x')$, $x' = (x_2, ..., x_d)$, and $\alpha >> 1$, $\ell >> 1$.
Then

$$\frac{(\nabla r \cdot \nabla d)}{r} = 2\alpha(x_1 - \ell)\frac{\partial_1 r}{r} + \frac{(\nabla_{x'} r \cdot \nabla_{x'} d)}{r} \geq C\alpha\delta_0 - C_1 > 0.$$

Case 3: local Carleman estimate near convex boundary point. Let

(i) Ω is convex near $(0, a)$ with $a > 0$.

(ii) $\partial_2 r(x) \geq 0$ for x near $(0, a)$.

Set $d(x) = \frac{x_1^2}{N^2} + \frac{x_2^2}{(a-\delta)^2}$ with $N >> 1$ and $0 < \delta << 1$. Then Carleman estimate (Theorem 3.1) holds.

We can apply to inverse source problem.

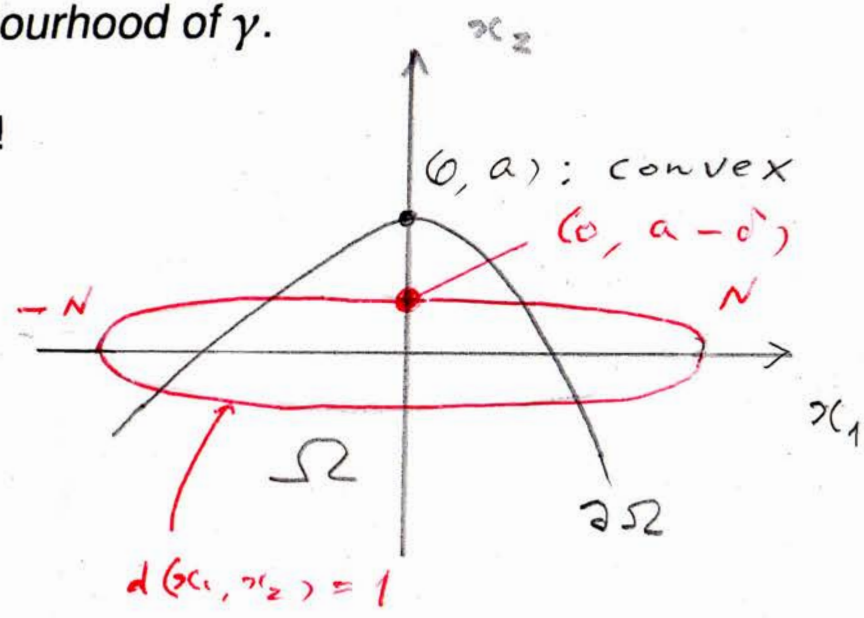
Set $\gamma = \partial\Omega \cap \{x; |x - (0, a)| < \varepsilon\}$ with small $\varepsilon > 0$.

Theorem 3.2 (local uniqueness near convex boundary point).

Let $r(x)\partial_t^2 u = \Delta u + R(x, t) f(x)$, $u(\cdot, 0) = \partial_t u(\cdot, 0) = 0$ and $R(\cdot, 0) > 0$ on $\overline{\Omega}$. Assume that $u = \partial_\nu u = 0$ on $\gamma \times (0, T)$. Then $f = 0$ in some neighbourhood of γ.

Remark. $\partial_2 r \geq 0$ is essential condition for uniqueness!

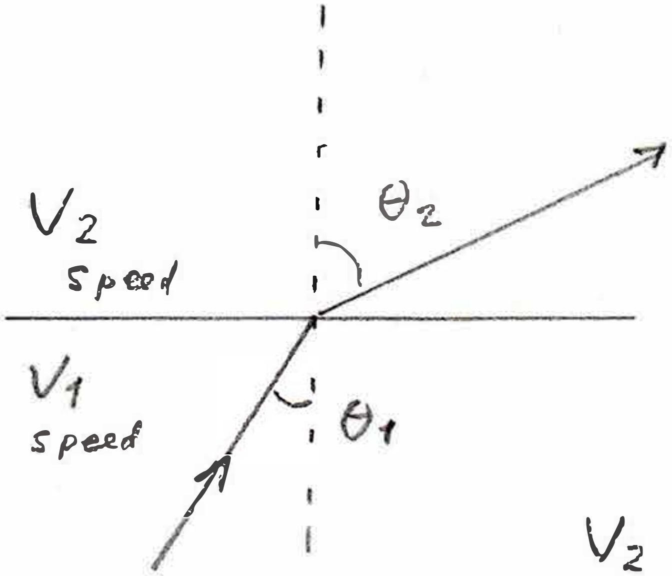
$\Longleftarrow$ Snell law



Remark. $\partial_2 r \geq 0$ means wave speed is increasing inward to Ω and is essential condition for uniqueness!

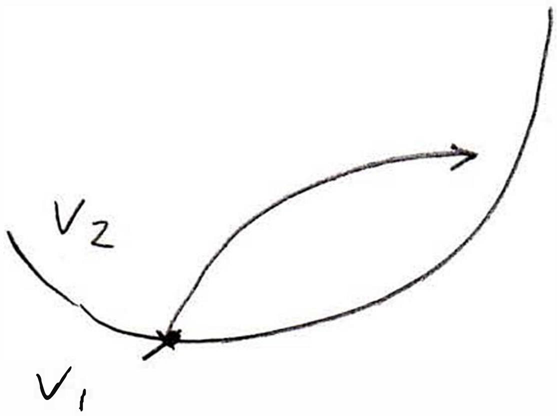
$\Longleftarrow$ Snell law

Snell law



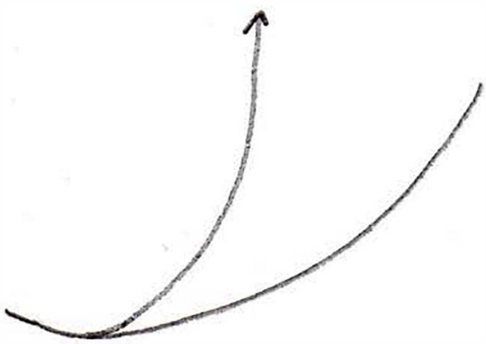
$$\frac{\sin \theta_2}{\sin \theta_1} = \frac{V_2}{V_1}$$

$$V_2 > V_1 \Longrightarrow \theta_2 > \theta_1$$



$V_2 > V_1$

signal :
easy to return



$V_2 < V_1$

possibly no return

How essential is $\frac{(\nabla r \cdot \nabla d)}{r} > -2\sigma_0$?

Recall $\varphi(x,t) = d(x) - \beta t^2$: weight of Carleman estimate and $(\partial_i \partial_j d)_{1 \le i,j \le d} \ge \sigma_0$ on $\overline{\Omega}$.

We have an example of hyperbolic equation rejecting Carleman estimate if

$$\frac{(\nabla r \cdot \nabla d)}{r} \le -4\sigma_0$$

$\Longleftarrow$ We can prove by Kumano-go (1963).

However his example has not been understood related to the impossibility of Carleman estimates in terms of

$$\frac{(\nabla r \cdot \nabla d)}{r} \le -4\sigma_0.$$

Example of hyperbolic equation not admitting Carleman estimates. Let $\Omega := \{\frac{1}{2} < |x| < 2\}$ and $r(x) := |x|^{-4}$. Then solution u to

$$\partial_t^2 u = |x|^4 \Delta u + \exists b(x,t)\partial_t u + \exists c(x,t)u \quad \text{in } \Omega \times (0,T)$$

satisfies $u(x,t) = 0$ for $1 \le |x| < 2$ and $t \in \mathbb{R}$, but $u(x,t) \ne 0$ for $\frac{1}{2} < |x| < 1$ and $t \in \mathbb{R}$.

This non-uniqueness implies no Carleman estimates.

§4. Inverse source problem for transmission wave equation

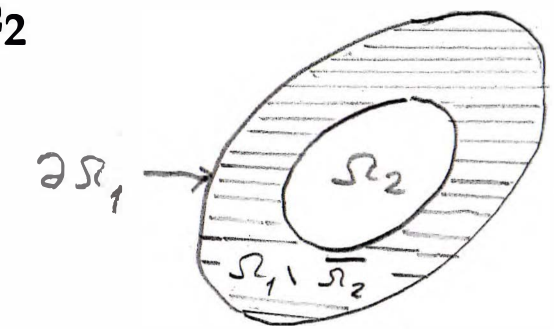
Discontinuous $a(x)$: important in geophysics $\Longleftarrow$ Mohorovicić discontinuity in Earth

Let $\overline{\Omega_2} \subset \Omega_1$.
Let $u_1 := u|_{\Omega_1 \setminus \Omega_2}$ and $u_2 := u|_{\Omega_2}$ satisfy

$$\partial_t^2 u = \mathrm{div}\,(a(x)\nabla u) + F(x,t) \quad \text{in } \Omega_1 \times (0,T),$$

where $a \in C^2(\overline{\Omega_2})$ and $a \in C^2(\overline{\Omega_1 \setminus \Omega_2})$

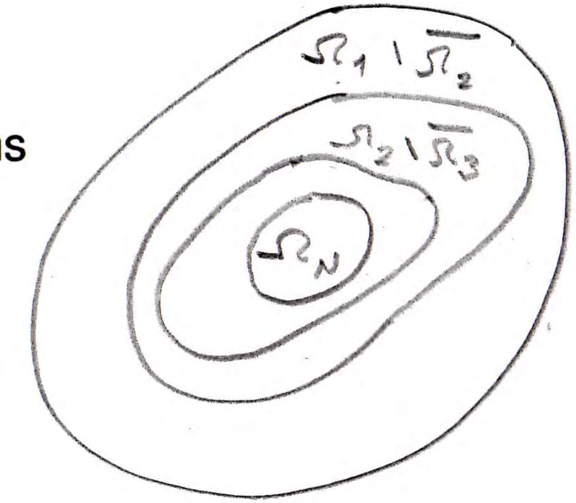
Transmission condition: $u_1 = u_2$ and $a_1 \partial_\nu u_1 = a_2 \partial_\nu u_2$ on $\partial\Omega_2$
Here ν: outward normal vector to $\partial\Omega_2$,
$a_1 := a|_{\Omega_1 \setminus \Omega_2}$ and $a_2 := a|_{\Omega_2}$.



Main results

Let $\Omega_N \subset \Omega_{N-1} \subset \cdots \Omega_2 \subset \Omega_1 \subset \mathbb{R}^n$: convex bounded domains such that $\overline{\Omega_k} \subset \Omega_{k-1}$ for $k = 2, ..., N$,

$a_k > 0$, $k = 1, 2, ..., N$ be constants.



Transmission wave equation:

$$\begin{cases} \partial_t^2 y_k = a_k \Delta y_k + R(x,t) f(x), \quad 0 < t < T, \\ \qquad x \in \Omega_k \setminus \overline{\Omega_{k+1}} \quad \text{if } k = 1, 2, ..., N-1, \quad x \in \Omega_N \quad \text{if } k = N, \\ y_{k-1} = y_k, \quad a_{k-1} \partial_\nu y_{k-1} = a_k \partial_\nu y_k \quad \text{on } \partial\Omega_k, \; k = 2, ..., N, \\ \text{zero Dirichler boundary condition on } \partial\Omega_1 \text{ and zero initial condition.} \end{cases}$$

Ω_k, a_k and $R(x,t)$: given.

Inverse source problem.

Determine $f(x)$, $x \in \Omega_1$ *by* $y_1|_{\partial\Omega_1 \times (0,T)}$, $\nabla y_1|_{\partial\Omega_1 \times (0,T)}$.

Theorem 4.1 (uniqueness).

Let $a_k > 0, \quad 1 \le k \le N$ *and* $R(x,0) \neq 0$ *for* $x \in \overline{\Omega_1}$, $R \in H^1(0,T;L^\infty(\Omega_1))$.
Then $\exists T > 0, \exists t_0 \in (0,T)$ *such that*
$y_1 = |\nabla y_1| = 0$ *on* $\partial\Omega_1 \times (0,T)$ *implies* $f = 0$ *in* Ω_1 *and* $y = 0$ *in* $\Omega_1 \times (0, T - t_0)$.

Henceforth set $y := y_k$ in $\Omega_k \setminus \Omega_{k+1}$, $k = 1...., N-1$, $y := y_N$ in Ω_N

Theorem 4.2 (global Lipschitz stability).

Let $0 < a_1 < a_2 < \cdots < a_N$ *and* $R(x,0) \neq 0$ *for* $x \in \overline{\Omega_1}$, $R \in H^2(0,T;L^\infty(\Omega_1))$. *Then*
$\exists T_0 > 0, \exists C > 0$ *such that*

$$C^{-1}\|\partial_\nu \partial_t y_1\|_{L^2(\partial\Omega_1\times(0,T_0))} \le \|f\|_{L^2(\Omega)} \le C\|\partial_\nu \partial_t y_1\|_{L^2(\partial\Omega_1\times(0,T_0))}$$

for any $f \in L^2(\Omega_1)$.

Remarks

- Bellassoued and Yamamoto (2017): global Carleman estimate for N-layer one-dimensional transmission wave equation
- Baudouin-Mercado-Osses (in Inverse Problems 2007):
  proved Global Lipschitz stability if $a_1 < a_2$ for 2-layer case.
  suggested that assumption $a_1 < a_2$ is essential in view of Snell's refraction law:
  $a_1 > a_2$ may cause trapping of rays so that the uniqueness might fail,
  but this is not correct.
- $a_1 < a_2$ is necessary for global Carleman estimate over the whole domain Ω_1.
  We do not use such global Carleman estimate here.
  $\Longrightarrow$ Assumption $a_1 < a_2$ is unnecessary for uniqueness.

Key

1. Carleman estimate $\Longrightarrow$ uniqueness (Theorem 1)
2. Observability inequality + uniqueness $\Longrightarrow$ global Lipschitz stability (Theorem 2)

§5. Recent result for Schrödinger equation

Imanuvilov-Yamamoto (2023): ArXiv 2212.13650

$$\sqrt{-1}\partial_t u = \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j u) + \sum_{k=1}^{d} b_k(x)\partial_k u + c(x,t)u,$$

where $a_{ij} = a_{ji}$: real-valued, $\in C^1(\overline{\Omega})$, uniformly elliptic,
$b_j \in W^{1,\infty}(\Omega)$, $c \in L^\infty(\Omega)$: complex-valued.
We consider only

Unique continuation. *Let* $\Gamma \subset \partial\Omega$ *be arbitrarily chosen subboundary and* $T > 0$ *be arbitrary.*
$u = \partial_\nu u = 0$ *on* $\Gamma \times (0,T)$ *implies* $u \equiv 0$ *in* $\Omega \times (0,T)$*?*

Remark. $\Gamma \supsetneqq$ (half of $\partial\Omega$) $\Longrightarrow$ Yes and observability inequality.

Carleman estimate directly does not produce unique continuation without geometric conditions on Γ.

Theorem 5.1. *Let* $\Gamma \subset \partial\Omega$, $T > 0$ *be arbitrarily chosen. Let* $u \in C([0,T]; H^2(\Omega))$ *satisfy*

$$\sqrt{-1}\partial_t u = \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j u) + \sum_{k=1}^{d} b_k(x)\partial_k u + c(x,t)u.$$

If $u = \partial_\nu u = 0$ *on* $\Gamma \times (0,T)$, *then* $u = 0$ *in* $\Omega \times (0,T)$.

Key of Proof. Enough to consider $\sqrt{-1}\partial_t u = \Delta u$. Then

$$w(x,t) := \int_0^T K(t,\tau)u(x,\tau)d\tau$$

where $K \in C^\infty([-1,1] \times [0,T])$ satisfies

$$\begin{cases} \sqrt{-1}\partial_\tau K - \partial_t^2 K(t,\tau) = 0, \quad -1 < t < 1,\ 0 < \tau < T, \\ K(-1,\tau) = \psi(\tau) \in C_0^\infty(0,T), \quad K(\cdot,0) = K(\cdot,T) = 0. \end{cases}$$

$\Longrightarrow$ We reduces to the unique continuation for elliptic equation $\partial_t^2 w + \Delta w = 0$.

Thank you very much!