Inverse scattering by corners and regular transmission eigenfunctions

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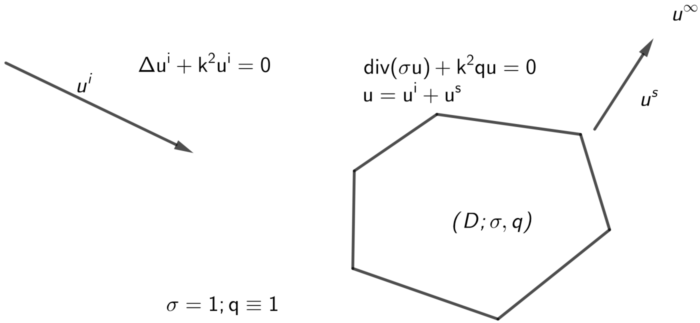
Inverse Scattering Problems

$$\Delta u^i + k^2 u^i = 0$$

$$\mathrm{div}(\sigma u) + k^2 q u = 0$$
$$u = u^i + u^s$$

$$\sigma = 1; q \equiv 1$$



Inverse Scattering Problems

Let $k \in \mathbb{R}_+$ be a wavenumber and u^i be an incident wave, i.e. an entire solution to the Helmholtz equation,

$$\Delta u^i + k^2 u^i = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{1}$$

Consider the following scattering problem,

$$\begin{cases} \operatorname{div}(\sigma \nabla u) + k^2 q u = 0 & \text{in } \mathbb{R}^n, \\ r^{\frac{n-1}{2}} (\partial_r - \mathrm{i}k)(u - u^i) \to 0 & \text{while } r \to \infty, \end{cases} \tag{2}$$

where $\sigma, q \in L^\infty(\mathbb{R}^n)$. The expansion at $+\infty$ holds,

$$u(x) = u^i(x) + \frac{e^{\mathrm{i}k|x|}}{|x|^{\frac{n-1}{2}}} u_\infty(\hat{x}; u^i) + \mathcal{O}(|x|^{-\frac{n}{2}}) \quad \text{as} \quad |x| \to +\infty,$$

where $u_\infty : \mathbb{S}^{n-1} \to \mathbb{C}$ is called the **far-field pattern**.
Inverse Problem: Recovery of σ, q from u_∞.

Recovery of $supp(q-1)$

Consider the scenario $\sigma \equiv 1$ in $\mathbb{R}^n$, let $D \Subset B_R \subset \mathbb{R}^n$ be a polytope, i.e. polygon in 2D or polyhedron in 3D. We assume $q = 1 + \phi\chi_D$ and $\phi(x_c) \neq 1$ at each corner.

Inverse Problem: Recovery of D from a single far-field pattern u_∞.

- Uniqueness: if $u_\infty = u'_\infty$ then $D = D'$.
  E. Blåsten and H. Liu, *Recovering piecewise constant refractive indices by a single far-field pattern*, Inverse Problems, **36(8)** (2020), 085005.
- Stability: $d_{\mathcal{H}}(D, D') \leq C(\ln|\ln\|u_\infty - u'_\infty\||)^\beta$.
  E. Blåsten and H. Liu, *On corners scattering stably and stable shape determination by a single far-field pattern*, Indiana Univ. Math. J., **70(3)**, (2021), pp.907-947.
- D is a smooth domain with high curvature points.
  E. Blåsten and H. Liu, *Scattering by curvatures, radiationless sources, transmission eigenfunctions and inverse scattering problems*, SIAM J. Math. Anal., **53(4)**, (2021), pp.3801-3837.

Recovery of $supp(\sigma - 1)$

Consider the scenario both the supports of $\sigma - 1$ and $q - 1$ are polytopes. Let D be the convex hull of $\text{supp}(\sigma - 1) \cup \text{supp}(q - 1)$.

- Uniqueness result
  F. Cakoni and J. Xiao, *On corner scattering for operators of divergence form and applications to inverse scattering*, Commun. Partial. Differ. Equ., **46(3)**, (2021), pp. 413–441.
- **Stability of polygons in** $\mathbb{R}^2$
  H. Liu and C.-H. Tsou, *Stable determination by a single measurement, scattering bound and regularity of transmission eigenfunction*, Calc. Var. Partial. Differ. Equ., **61** (2022), No. 91.

Stability of Polygons - Assumptions

We suppose the following *admissible* assumptions of the medium scatterer $(D; \sigma, q)$.

1. $\sigma := 1 + (\gamma - 1)\chi_D$ in $\mathbb{R}^2$.
2. $D \Subset B_R$ is a convex polygon with certain $R > 0$.
3. $\gamma \in \mathbb{R}$ satisfying $0 < \gamma_m \leq \gamma \leq \gamma_M$.
4. For any vertex x_c of D, the opening angle a satisfies $0 < a_m \leq a \leq a_M < \pi$.
5. The length of each edge of D is at least $l > 0$.
6. $\text{supp}(q - 1) \Subset D$, i.e. $q \equiv 1$ in $\mathbb{R}^2 \setminus \overline{D}$.
7. $\|q\|_{L^\infty(\mathbb{R}^2)} \leq \mathcal{Q}$ where $\mathcal{Q} > 0$ is a constant.

The parameters $k_m, k_M, a_m, a_M, l, \mathcal{Q}$ are called the a-priori data.
Let $u^i \in H^2_{loc}(\mathbb{R}^2)$ be an incident wave. We denote by $S > 0$ the *amplitude* of the incident wave u^i, which is defined by $\|u^i\|_{H^2(B_{2R})} \leq S$

Stability Estimation

- Let $k \in \mathbb{R}_+$ and $u^i \in H^2_{loc}(\mathbb{R}^2)$ be an incident wave.
- Let $(D; \sigma, q)$ and $(D'; \sigma', q')$ be admissible scatterers.
- $d_{\mathcal{H}}(D, D')$ designs the Hausdorff distance between D and D'.
- Let u and u' be the total waves respectively corresponding to $(D; \sigma, q)$ and $(D'; \sigma', q')$.
- Suppose that u and u' admit the non-degenerate corner singularities.

Theorem 1 (Liu and Tsou 22')

If

$$\|u_\infty - u'_\infty\|_{L^2(\mathbb{S}^1)} \leq \varepsilon,$$

then the stability estimation holds

$$d_{\mathcal{H}}(D, D') \leq C \left(\ln|\ln\varepsilon|\right)^{-\beta}, \tag{3}$$

where $\beta > 0$ *and*

$$C = \widetilde{C}\left(1 + \frac{S}{K_m}\right)^{\widetilde{\beta}}.$$

Ingredients of the Proof

- **Corner Singularity**: Genetically, u, u' don't admit the H^2 regularity near each edge and ∇u, $\nabla u'$ blow up near each corner. Key observation.
- **Propagation of Smallness**:
  1. From the far-field to the near-field: Quantitative Rellich's theorem.
  2. Propagation from near-field to the scatterer: Quantitative unique continuation property.

  Estimation of $u - u'$ near $D \cup D'$.
- **Micro-local Analysis**: convex polygons $\Rightarrow d_{\mathcal{H}}(D, D') = |x_c - x'|$ with x_c a corner of D and $x' \in \partial D'$. Reasoning in the phase space of CGO solutions defined near the corner x_c. Link between the singularities and the estimations of $u - u'$.

Corner Singularity

Theorem 2 (Grisvard 85', Dauge and Nicaise 89')

Let $u \in H^1_{loc}(\mathbb{R}^2)$ be the solution to (2) with $(D; \sigma, q)$ satisfying the admissible assumptions. We denote by $\mathcal{S}_D$ the set of vertices of D. Then the following decomposition holds,

$$u = u_{reg} + u_{sing} = u_{reg} + \sum_{x_i \in \mathcal{S}_D} K_i r^{\eta_i} \phi_i(\theta) \zeta_i. \tag{4}$$

- $u_{reg} \in PH^2(B_R)$ and satisfies $\|u_{reg}\|_{H^2(D)} \leq C\|u\|_{H^1(B_R)}$ with C depending only on the *a-priori data*.
- The exponent $\eta_i \in (0, 1)$ depends explicitly on the parameter γ and the opening angle a at the vertex x_i.
- $\phi_i(\theta) = \cos(\eta_i \theta + \Phi_{i,\pm})$.
- The coefficient K_i depends linearly on the incident wave u_i. we assume genetically that $K_i \neq 0$ for all vertex x_i and set $K_m := \min_{x_i \in \mathcal{S}_D} |K_i|$.

Propagation of smallness: far-field to near-field

- Let $w^s \in H^2_{loc}(\mathbb{R}^2)$ be a solution to (1) in $\mathbb{R}^2 \setminus B_R$.
- w^s satisfy the Sommerfeld radiation condition at infinity.
- *a-priori* bound $\|w^s\|_{L^2(B_{2R} \setminus B_R)} \leq \mathcal{S}$.

Proposition 3

If the far-field pattern $\|w^s_\infty\|_{L^2(\mathbb{S}^1)} = \varepsilon$ *is small enough, then*

$$\|w^s\|_{H^p(\mathcal{A})} \leq C \max(\varepsilon, \mathcal{S} e^{-c\sqrt{\ln(\mathcal{S}/\varepsilon)}}). \tag{5}$$

Proof: Elliptic interior regularity and estimations of Hankel functions.

Propagation of smallness: from near-field to the scatterer

Let u be a solution to $\operatorname{div}(\sigma\nabla u) + k^2 qu = 0$ in a bounded domain, we use the unique continuation property[1] to estimate $\|u\|_N$ in Ω from the knowledge of $(u|_{\Gamma_0}, \partial_\nu u|_{\Gamma_0})$.

1. Three-spheres inequality.
2. Iteration of three spheres inequality from boundary to interior.
3. Extension of the solution near Γ_0.
4. Conclusion with interior/global estimations.

Lemma 4 (Three-sphere Inequality, Alessandrini 09')

Let $0 < r_1 < r_2 < r_3 < R$, and $w \in H^1_{loc}(B_R)$ be a solution to (1) in B_R. Then there exists $\tilde{\alpha} \in (0,1)$, which depends only on r_2/r_1 and r_3/r_2, such that

$$\|w\|_{L^\infty(B_{r_2})} \leq C\|w\|^{1-\tilde{\alpha}}_{L^\infty(B_{r_3})}\|w\|^{\tilde{\alpha}}_{L^\infty(B_{r_1})}.$$

[1] G. Alessandrini et al. *The stability for the Cauchy problem for elliptic equations*, Inverse Problems, **25**, (2009), 123004

Propagation of smallness

- Let $u, u' \in H^1_{loc}(\mathbb{R}^2)$ be the solutions of the scattering problems (2) under the assumptions of Theorem 1.
- Let x_c be a vertex of Q, which is the convex hull of $D \cup D'$.
- $u - u'$ is of class $\mathcal{C}^{\alpha_0}$ in $B_{2R} \setminus \overline{Q}$.
- The function $x \mapsto |x - x_c| \nabla(u - u')(x)$ is of class $\mathcal{C}^{\alpha_1}$ in $B_{2R} \setminus \overline{Q}$.

Proposition 5

If $\|u_\infty - u'_\infty\|_{L^2(\mathbb{S}^1)} \leq \varepsilon$ for ε small enough, it holds that

$$|u(x) - u'(x)| \leq \widetilde{C_0} T_0 \left(\ln \ln \frac{S}{\varepsilon} \right)^{-\alpha_0}, \tag{6}$$

$$|\nabla(u - u')(x)| \leq \frac{\widetilde{C_1} T_1}{\operatorname{dist}(x, \partial Q)} \left(\ln \ln \frac{S}{\varepsilon} \right)^{-\alpha_1}, \tag{7}$$

for $x \in B_{3R/2} \setminus \overline{Q}$.

Micro-local Analysis

Main Steps

1. Establish an integral identity.
2. Choose adaptable complex geometric optical solutions (CGO)

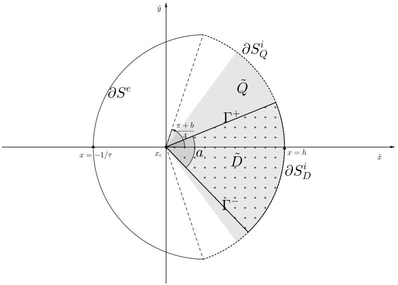
$$u_0(x) = e^{\rho(\tau)\cdot x}(1 + \psi(x)).$$

3. Estimate of each terms in the integral identity.
4. Delicate balancing the parameter τ in the phase of the CGO solutions.
5. Conclusion by deducing the stability result.

CGO solutions

Up to a rigid motion, we choose the coordination $x_c = 0$, $D \subset \{x > 0\}$.
Let $\tau > 0$, we set $\rho = \rho(\tau) := \tau(-\hat{x} + i\hat{y}) \in \mathbb{C}^2$. For all $x \in \mathbb{R}^2$,

$$u_0(x) = e^{\rho\cdot(x-x_c)}. \tag{8}$$



Integral identity

$$(1-\gamma)\int_{\Gamma^\pm} u_0 \partial_\nu u \, ds =$$

$$\int_{\partial S_Q^i \cup \partial S^e} (u-u')\partial_\nu u_0 - u_0 \partial_\nu (u-u')\, ds - k^2 \int_{\widetilde{D}^e} (u-u') u_0 \, dx - \frac{k^2 q}{\gamma} \int_{\widetilde{D}} u_0 u \, dx$$

Estimations

Corner singularity decomposition near a vertex x_c:

$$u = u_{sing}\zeta + u_{reg} \quad \text{with} \quad u_{sing}(r,\theta) = Kr^{\eta}\cos(\eta\theta + \Phi).$$

Proposition 6 (upper bound)

Let $\tau > 0$, u_0 be a CGO solution defined by (8). Then, the estimation holds,

$$\begin{aligned} C\Big|\int_{\Gamma_\infty^{\pm}} u_0 \partial_\nu u_{sing}\, ds\Big| &\le |K|\tau^{-\eta} e^{-\alpha'\tau h/2} + he^{-\alpha'\tau h}\|u_{reg}\|_{H^2(\widetilde{D})} \\ &+ he^{-\alpha'\tau h}(\|\partial_\nu(u-u')\|_{L^\infty(\partial S_Q^i)} + \tau\|u-u'\|_{L^\infty(\partial S_Q^i)}) + \tau^{-1}\|u_{reg}\|_{H^2(\widetilde{D})} \\ &+ h(\|\partial_\nu(u-u')\|_{L^\infty(\partial S^e)} + \tau\|u-u'\|_{L^\infty(\partial S^e)}) \\ &+ h^2\|u-u'\|_{L^\infty(\widetilde{D}^e)} + (\tau^{-1} + he^{-\alpha'\tau h})\|u\|_{H^1(\widetilde{D})}, \end{aligned} \tag{9}$$

where C depends only a-priori data.

Estimations

Corner singularity decomposition near a vertex x_c:

$$u = u_{sing}\zeta + u_{reg} \quad \text{with} \quad u_{sing}(r,\theta) = Kr^{\eta}\cos(\eta\theta + \Phi).$$

Proposition 7 (lower bound)

Let $\tau > 0$, u_0 be a CGO solution defined by (8). Then, the estimation holds,

$$\left| \int_{\Gamma^{\pm}_{\infty}} u_0 \partial_\nu u_{sing} d\sigma \right| = K\Gamma(\eta) \left| \phi'(\theta^+) e^{ia\eta} - \phi'(\theta^-) \right| \tau^{-\eta} \geq K\Gamma(\eta) \sin(a\eta) \tau^{-\eta}, \tag{10}$$

where $\theta^{\pm}$ signify the arguments of the vectors along $\Gamma^{\pm}$.

Proof of the Stability

Using the corner singularity theorem and the unique continuation property to estimate the L^∞ or H^2 norms in the right-hand-side of (9), the inequalities (9),(10) imply

$$C|K|\tau^{-\eta} \leq |K|\tau^{-\eta}e^{-\alpha'\tau h/2} + (|K|h^{\eta-1} + Sh^{\eta'-1} + S + S\tau)he^{-\alpha'\tau h} + (|K| + S)h\tau\delta(\varepsilon) + (|K| + S)h^2\delta(\varepsilon) + S\tau^{-1} + She^{-\alpha'\tau h}.$$

Calculations and trivial inequalities lead to

$$C \leq (1 + \frac{S}{|K|})(h^{-1}\tau^{\eta-1} + h\tau^{\eta+1}\delta(\varepsilon)).$$

We next determine a minimum modulo constants of the right hand side. Set $\tau = \tau_e$ with

$$\tau_e = h^{-1}\delta(\varepsilon)^{-1/2}.$$

Solving for h, it gives

$$h \leq C(1 + \frac{S}{|K|})^{\frac{1}{\eta}} \left(\ln\ln\frac{S}{\varepsilon}\right)^{\frac{\eta m(\eta-1)}{2\eta}}.$$

Corner Always Scatter

The presence of the corner singularities induces non zero far-field patterns.

- Let D be a Lipschitz domain in $\mathbb{R}^2$, not necessarily a convex polygon.
- ∂D admits a convex polygonal point.
- Let u be the solution to the scattering problem with the far-field pattern u_∞.

Theorem 8 (Liu and Tsou 22')

Under the assumptions of Theorem 1, it holds

$$\|u_\infty\|_{L^2(\mathbb{S}^1)} \geq \frac{S}{\exp\exp\left(C(1+\frac{S}{|K|})^{\frac{2}{\eta(1-\eta)}}\right)}. \tag{11}$$

Proof: Taking $D' = \emptyset$ and $q' \equiv 1$ in $\mathbb{R}^2$, then apply the stability estimate result.

Transmission Eigenvalue Problems

If $u_\infty = 0$ on $\mathbb{S}^1$ occurs, the following equation admits a nontrivial solution.

$$\begin{cases} \mathrm{div}(\sigma \nabla u) + k^2 q u = 0 & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ u = v, \quad \sigma \partial_\nu u = \partial_\nu v & \text{on } \partial D. \end{cases} \tag{12}$$

The solution (u, v) is called the **transmission eigenfunction** associated to the **transmission eigenvalue** k.

Herglotz wave approximation[2]: For any $\varepsilon \ll 1$, there exists $g_\varepsilon \in L^2(\mathbb{S}^1)$ such that

$$\|v_{g_\varepsilon} - v\|_{H^1(D)} \leq \varepsilon, \quad v_{g_\varepsilon}(x) := \int_{\mathbb{S}^1} e^{\mathrm{i}kx\cdot d} g_\varepsilon(d)\, ds(d), \tag{13}$$

[2] N. Weck, *Approximation by Herglotz wave functions*, Math. Methods Appl. Sci., **27(2)**, (2004), pp.155-162

Implications of polytope supports

Let (u, v) the transmission eigenfunction associated to the eigenvalue k. If $\sigma \equiv 1$ and $\mathrm{supp}(q-1)$ is a polytope in $\mathbb{R}^n$, $n = 2, 3$.

- v cannot extended to an entire solution to (1) in $\mathbb{R}^n$.
  E. Blåsten, L. Päivärinta and J. Sylvester, *Corners always scatter*, Comm. Math. Phys., **331** (2014), pp. 725–753.
- If v can be approximated by Herglotz wave functions, then $v(x_c) = 0$ at each corner.
  E. Blåsten and H. Liu, *On vanishing near corners of transmission eigenfunctions*, J. Functional Analysis, **273** (2017), pp. 3616–3632. Addendum is available at arXiv:1710.08089.

Regularity Result

- Let (D, σ, q) be a convex polygonal scatter satisfying the assumptions in Theorem 1.
- Let $u, v \in H^1(D)$ be a nontrivial eigenfunctions of the transmission eigenvalue problem (12).

Theorem 9 (Liu and Tsou 21')

1. *If v can be extended outside D to be an entire solution to (1), then $u \in H^2(D)$.*
2. *If v can be approximated by Herglotz wave functions and those functions are uniformly bounded, then it holds*

$$\lim_{x \neq x' \in D, x, x' \to x_c} \frac{|u(x) - u(x')|}{|x - x'|^{\eta}} = 0. \tag{14}$$

Conclusions and Perspectives

Conclusions

- Stability estimates for polygonal scatterer.
- Application of micro-local analysis and corner singularity decomposition to the study of inverse scattering problems.
- Implication to the transmission eigenvalue problems.

Perspectives

- Combination of the recovery of the conductivity σ and the potential q.
- Extension three dimensional polyhedrons, variable or anisotropic conductivity σ.

THANKS FOR YOUR ATTENTION !