Determining a Parabolic System by its Boundary Data with Biological Applications

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Joint work with Hongyu Liu

Introduction

Consider the following coupled nonlinear system of parabolic equations:

$$\begin{cases} \partial_t u(x,t) = F(x,t,u,v,\nabla u,\nabla v,\Delta u,\Delta v) & \text{in } \Omega \times (0,T), \\ \partial_t v(x,t) = G(x,t,u,v,\nabla u,\nabla v,\Delta u,\Delta v) & \text{in } \Omega \times (0,T), \\ u,v \geq 0 & \text{in } \bar{\Omega} \times [0,T), \end{cases} \tag{1}$$

$\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain,
$T \in (0,\infty]$,
$F(x,t,p_1,q_1,p_2,q_2,p_3,q_3) : \Omega \times (0,T) \times \mathbb{R}^{2n+4} \to \mathbb{R}$,
$G(x,t,p_1,q_1,p_2,q_2,p_3,q_3) : \Omega \times (0,T) \times \mathbb{R}^{2n+4} \to \mathbb{R}$ are real-valued functions with respect to p_i and q_i, $i = 1,2,3$.

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Inverse Problem

Define the measurement map $\mathcal{M}^{+}_{F,G}$:

$$\mathcal{M}^{+}_{F,G}(u|_{\Sigma}) = \partial_\nu u(x,t)\big|_{\Sigma}, \quad \Sigma := \partial\Omega \times (0,T)$$

where " $+$ " signifies that the boundary data of u (or, v) are associated with the non-negative solutions of the coupled parabolic system.

The inverse problem mentioned above can be formulated as

$$\mathcal{M}^{+}_{F,G} \longrightarrow F, G.$$

Inverse Problem (Unique Identifiability)

Can one establish the following one-to-one correspondence for two configurations (F^j, G^j), $j = 1, 2$:

$$\mathcal{M}^+_{F^1,G^1} = \mathcal{M}^+_{F^2,G^2} \quad \text{if and only if} \quad (F^1, G^1) = (F^2, G^2).$$

Our main result is given, formally, as follows: In this paper, we aim to prove, in formal terms, the following theorem.

Theorem

Let $\mathcal{M}^+_{F^j,G^j}$, $j = 1, 2$, be the measurement map associated to (1). Assume $F^j, G^j \in \mathcal{A}$, where $\mathcal{A}$ is a certain admissible class. Suppose

$$\mathcal{M}^+_{F^1,G^1}(u|_{\partial\Omega}) = \mathcal{M}^+_{F^2,G^2}(u|_{\partial\Omega}) \quad \textit{for all } u|_{\partial\Omega} \in \mathcal{S},$$

where $\mathcal{S}$ is a properly chosen function space on Σ. Then

$$(F^1, G^1) = (F^2, G^2).$$

Motivation

- Fokker-Planck equation:
  $u_t = L^* u,\ Lv = tr(A\Delta\varphi) + \langle b, \nabla\varphi\rangle + c\varphi$
- Mean field games:
  $-u_t - \Delta u + H(x, \nabla u) = F(x, t, m),$
  $m_t - \Delta m - \nabla \cdot (m \nabla_p H(x, \nabla u)) = 0$
- Gas-liquid interaction problems:
  $u_t - D_1 \Delta u = f_1(x, u, v),\ v_t - D_2 \Delta v = f_2(x, u, v),$
  $f_i(x, u, v) = -\sigma_i u^m v^n + q_i(x),\ m, n \geq 1, q_i(x) \geq 0, i = 1, 2$
- Belousov-Zhabotinskii oregonator model:
  $u_t - D_1 \Delta u = u(a - bu - cv),\ v_t - D_2 \Delta v = -c' uv$
- Volterra-Lotka model: $u_t - D_1 \Delta u = u(a_1 - b_1 u \pm c_1 v),$
  $v_t - D_2 \Delta v = v(a_2 \pm b_2 u - c_2 v)$
- Epidemic Kermack-McKendrick equation:
  $u_t - D_1 \Delta u = -a_1 u - b_1 u \int_\Omega K(x, \xi) v(t, \xi) d\xi,$
  $v_t - D_2 \Delta v = -a_2 u - b_2 u \int_\Omega K(x, \xi) v(t, \xi) d\xi$

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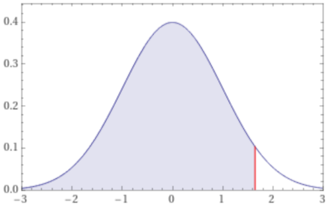

Source: Wolfram MathWorld

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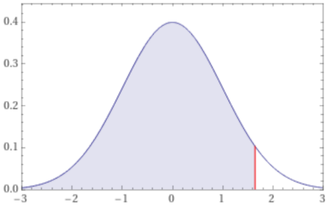

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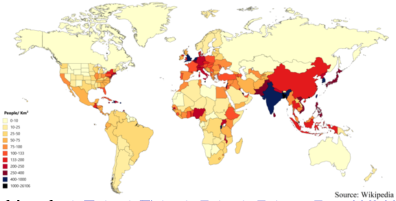
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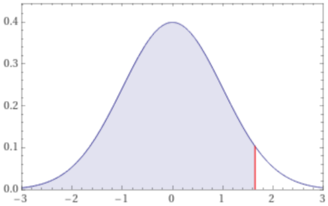

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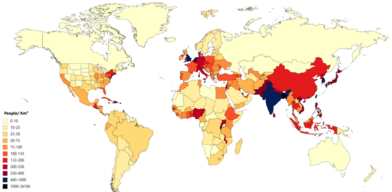
They may model different physical phenomena, but for all of them, $u, v \geq 0$


Source: Wikipedia

Motivation

Also for equations where solutions are not necessarily non-negative, but non-negative solutions have additional properties

- Burgers' equation: $\boldsymbol{u}_t + \boldsymbol{u}\boldsymbol{u}_x = \nu \Delta \boldsymbol{u} + \boldsymbol{f}(x, t)$
- Allen-Cahn equation: $\phi_t = \epsilon^2 \Delta \phi - \frac{1}{\epsilon^2} W'(\phi)$
- Fisher-KPP equation: $u_t - D\Delta u = F(u)$
- Nonlinear Schrödinger equation: $i\psi_t = -\frac{1}{2}\Delta \boldsymbol{\psi} + \kappa |\boldsymbol{\psi}|^2 \boldsymbol{\psi}$
- Hamilton-Jacobi equation: $S_t = -H(x, \nabla S, t)$

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3. We used the classical high-order linearisation technique around a pair of trivial solutions $(0, 0)$:

$$u(x, t; \varepsilon) = \sum_{l=1}^{\infty} \varepsilon^l f_l \quad \text{on } \Sigma \quad \text{for } f_1 > 0.$$

Here, $f_2(x, t)$ may possibly be positive or negative at different x, t, but for all small positive ε, the positivity of f_1 ensures that $u(x, t; \varepsilon) > 0$ on Σ.

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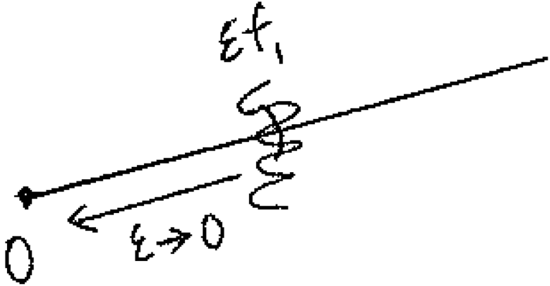
4. Our measurement map only involves u, and no information is required for v.

Main Idea

Previous results:



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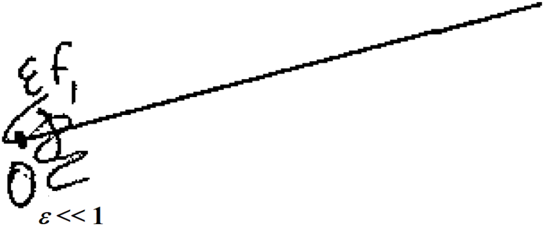

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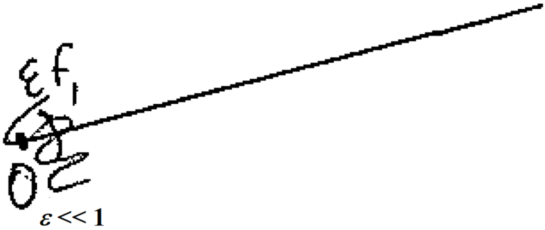
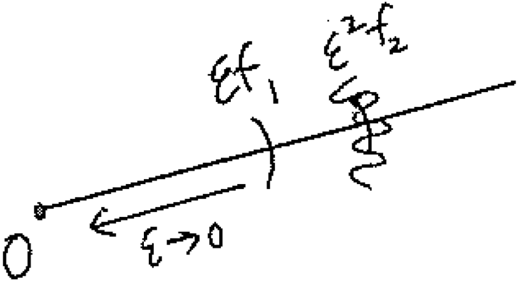


Solution is not non-negative/positive!

Main Idea

Our method with higher order variation:



$f_1 > 0$

Mathematical Setup

$$
\begin{cases}
\partial_t u(x,t) - \mu \Delta u(x,t) = F(x,t,u,v) & \text{in } Q, \\
\partial_t v(x,t) - \nu \Delta v(x,t) = G(x,t,u,v) & \text{in } Q, \\
u, v \geq 0 & \text{in } Q, \\
u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 & \text{in } \Omega, \\
u = f \geq 0, \quad v = g \geq 0 & \text{on } \Sigma
\end{cases} \tag{2}
$$

where $Q := \Omega \times (0,T)$ for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $\Sigma := \partial\Omega \times (0,T)$, $T \in (0,\infty]$.

Here, $\mu, \nu > 0$ are positive constants, which may represent the viscosity coefficient in the Burgers' equation, thickness of the layer between two phases in the Allen-Cahn equation, additive noise in the Fokker-Planck equation, or diffusion coefficients in population or chemical models.

Observe that $(0,0)$ is a solution to the problem.

Mathematical Setup

The functions $F(x,t,p,q), G(x,t,p,q) : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are analytic with respect to p and q, and are of the form

$$F(x,t,p,q) := \sum_{\substack{m,n\geq 0 \\ m+n\geq 3}}^{\infty} \alpha_{mn}(x,t)p^m q^n$$

and

$$G(x,t,p,q) := \sum_{\substack{m,n\geq 0 \\ m+n\geq 1}}^{\infty} \beta_{mn}(x,t)p^m q^n,$$

such that

$$\beta_{01}(x,t) \leq 0. \tag{3}$$

Mathematical Setup (Inverse Problem)

We want to determine the coefficients α_{mn} and β_{mn}, using knowledge of u at the boundary of some bounded domain Σ.
We introduce the measurement map $\mathcal{M}^{+}_{F,G}$

$$\mathcal{M}^{+}_{F,G}(u|_{\Sigma}) = \partial_{\nu} u(x,t)\big|_{\Sigma}\,.$$

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Physically, this means that we assume that all agents follow the parabolic system (i.e. on a macro scale, they follow laws of nature), and the observer only knows the value functions of the agents at the boundary of some chosen domain. The main goal is to recover some information regarding the environment, such as source functions or forcing functions.

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We measure/observe the space-time boundary data of u, from which we can determine the interacting functions F and G over the space-time domain Q.

Mathematical Setup (Inverse Problem - Unique Identifiability)

In particular, we are mainly concerned with the unique identifiability issue, which asks whether one can establish the following one-to-one correspondence:

$$\mathcal{M}^{+}_{F^1,G^1} = \mathcal{M}^{+}_{F^2,G^2} \quad \text{if and only if} \quad (F^1, G^1) = (F^2, G^2)$$

two configurations (F^j, G^j), $j = 1, 2$.

Mathematical Setup (Classic Forward Problem)

Theorem

Suppose that the first derivatives of F, G are continuous with respect to x, t, u, v. For $\alpha \in (0,1)$, assume $u_0, v_0 \in C^{2+\alpha}(\bar{\Omega})$, $f, g \in C^{2+\alpha, 1+\alpha/2}(\bar{\Sigma})$ such that $u_0, v_0, f, g \geq 0$ with the compatibility conditions

$u_0(x) = f(x,0)$ *and* $f_t(x,0) = \mu \Delta u_0(x) + F(x, 0, u_0(x), v_0(x))$ *on* Σ

and

$v_0(x) = g(x,0)$ *and* $g_t(x,0) = \nu \Delta v_0(x) + G(x, 0, u_0(x), v_0(x))$ *on* Σ.

Then, the system (2) admits a unique non-negative solution

$$(u, v) \in [C^{2+\alpha, 1+\alpha/2}(\bar{Q})]^2.$$

Mathematical Setup (Admissible Class)

Suppose F and G are analytic, and we impose the following condition a priori on F and G.

Definition

$U(x, t, p, q) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is admissible, denoted by $U \in \mathcal{A}$, if:

1. The map $z \mapsto U(\cdot, \cdot, p, q)$ is holomorphic with value in $C^{2+\alpha, 1+\alpha/2}(\bar{Q})$ for some $\alpha \in (0, 1)$,
2. $U(x, t, 0, 0) = 0$ for all $(x, t) \in Q$.

If U satisfies these conditions, U has a power series expansion

$$U(x, z) = \sum_{m,n=1}^{\infty} U^{(m,n)}(x) \frac{p^m q^n}{(m+n)!},$$

where $U^{(m,n)}(x, t) = \frac{\partial^m}{\partial p^m} \frac{\partial^n}{\partial q^n} U(x, t, 0) \in C^{2+\alpha, 1+\alpha/2}(\bar{Q})$.

Main Result

Theorem

Let $\mathcal{M}^+_{F^j,G^j}$ be the measurement map associated to (2) for $j = 1, 2$. Assume $F^j, G^j \in \mathcal{A}$ such that (3) holds. Suppose, for any

$$u(x,t) = \sum_{l=1}^{\infty} \varepsilon^l f_l \quad \text{on } \Sigma$$

where $f_l \in C^{2+\alpha,1+\alpha/2}(\bar{\Sigma})$ with $|\varepsilon|$ small enough such that $f_1(x,0) = u_0(x)$ and $f_l(x,0) = 0$ for $l \geq 2$, one has

$$\mathcal{M}^+_{F^1,G^1}(u|_{\partial\Omega}) = \mathcal{M}^+_{F^2,G^2}(u|_{\partial\Omega}) \quad \text{for all } u|_{\partial\Omega} \in \mathcal{S} := C^{2+\alpha,1+\alpha/2}(\bar{\Sigma}).$$

Fix $m = 1, \ldots M$, $M < \infty$.

1. *For $m = 1$, if $\beta_{11} = \beta_{02} \equiv 0$, and β_{01}, β_{20} are known, fixed, then*

$$\beta^1_{10}(x,t) = \beta^2_{10}(x,t) \quad \text{in } Q. \tag{4}$$

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2. $m \geq 2$: *if $\beta_{10}, \alpha_{m_2 n_2}$ are known, fixed for all $m_2 + n_2 = m + 1$, and*

$$\alpha_{m_1 n_1} \equiv 0 \quad \text{for all } 2 \leq m_1 + n_1 \leq m, m_1 \neq m,$$

then $$\alpha^1_{m0}(x,t) = \alpha^2_{m0}(x,t) \quad \text{in } Q.$$

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Suppose, for any

$$u(x,t) = \sum_{l=1}^{\infty} \varepsilon^l f_l \quad \text{on } \Sigma,$$

one has

$$\mathcal{M}^+_{F^1,G^1}(u|_{\partial\Omega}) = \mathcal{M}^+_{F^2,G^2}(u|_{\partial\Omega}).$$

3. $m \geq 2$: *suppose* $\alpha_{mn}, \beta_{m_2,n_2}$ *are known, fixed for* $m_2 + n_2 = m+1$ *or* $m_2 + n_2 \leq 1$,

$$\beta_{m_1 n_1} \equiv 0 \quad \text{for all } 2 \leq m_1 + n_1 \leq m, m_1 \neq m,$$

If either $\alpha_{m_1 n_1} \equiv 0$ *for all* $2 \leq m_1 + n_1 \leq m$ *or* $\beta_{10} \equiv 0$,
then $\beta^1_{m0}(x,t) = \beta^2_{m0}(x,t)$ *in* Q.

Main Result (Remark)

Observe that the recovery of these coefficients is not simultaneous. On the other hand, as long as the assumptions are satisfied for some $m \geq 2$, it is possible to obtain that the results of (2) and (3) separately by choosing the same $u(x,t) = \sum_{l=1}^{m+1} \varepsilon^l f_l$ on Σ.

Proof (Linearisation)

Let

$$u(x,t;\varepsilon) = \sum_{l=1}^{\infty} \varepsilon^l f_l \quad \text{on } \Sigma,$$

where

$$f_l \in C^{2+\alpha,1+\alpha/2}(\bar{\Sigma})$$

with $|\varepsilon|$ small enough, satisfying

$$f_1(x,0) = u_0(x) \quad \text{and} \quad f_l(x,0) = 0 \text{ for } l \geq 2.$$

Assume

$$f_1(x,t) > 0 \quad \forall x \in \Omega, t \in (0,T),$$

so that

$$\text{for all small } \varepsilon > 0, \quad u(x,t;\varepsilon) > 0 \text{ on } \Sigma.$$

Then, by the classical result for the forward problem, there exists a unique solution $(u(x,t;\varepsilon), v(x,t;\varepsilon))$ of (2).

Proof (First Order Linearisation)

Let $(u(x,t;0), v(x,t;0)) = (0,0)$ be the solution of (2) when $\varepsilon = 0$.
Define

$$u^{(1)} := \partial_\varepsilon u\big|_{\varepsilon=0} = \lim_{\varepsilon\to 0} \frac{u(x,t;\varepsilon) - u(x,t;0)}{\varepsilon},$$

$$v^{(1)} := \partial_\varepsilon v\big|_{\varepsilon=0} = \lim_{\varepsilon\to 0} \frac{v(x,t;\varepsilon) - v(x,t;0)}{\varepsilon},$$

and consider the new system associated to $(u^{(1)}, v^{(1)})$:

$$\begin{cases} \partial_t u^{(1)}(x,t) - \mu\Delta u^{(1)}(x,t) = 0 & \text{in } Q, \\ \partial_t v^{(1)}(x,t) - \nu\Delta v^{(1)}(x,t) = & \\ \qquad\qquad \beta_{10}(x,t)u^{(1)}(x,t) + \beta_{01}(x,t)v^{(1)}(x,t) & \text{in } Q, \\ u^{(1)}(x,0) = u_0(x) \geq 0, \quad v^{(1)}(x,0) = v_0(x) \geq 0 & \text{in } \Omega, \\ u^{(1)}(x,t) = f_1(x,t) > 0, \quad v^{(1)}(x,t) = g(x,t) \geq 0 & \text{on } \Sigma. \end{cases} \tag{10}$$

First Order Linearisation – u

$$\begin{cases} \partial_t u^{(1)}(x,t) - \mu\Delta u^{(1)}(x,t) = 0 & \text{in } Q, \\ \partial_t v^{(1)}(x,t) - \nu\Delta v^{(1)}(x,t) = \\ \qquad\qquad \beta_{10}(x,t)u^{(1)}(x,t) + \beta_{01}(x,t)v^{(1)}(x,t) & \text{in } Q, \\ u^{(1)}(x,0) = u_0(x) \geq 0, \quad v^{(1)}(x,0) = v_0(x) \geq 0 & \text{in } \Omega, \\ u^{(1)}(x,t) = f_1(x,t) > 0, \quad v^{(1)}(x,t) = g(x,t) \geq 0 & \text{on } \Sigma. \end{cases} \tag{11}$$

Then, $u^{(1)} \in C^{2+\alpha,1+\alpha/2}(\bar{Q})$ is the strictly positive solution of the heat equation,

First Order Linearisation – u

$$
\begin{cases}
\partial_t u^{(1)}(x,t) - \mu\Delta u^{(1)}(x,t) = 0 & \text{in } Q,\\
\partial_t v^{(1)}(x,t) - \nu\Delta v^{(1)}(x,t) = \\
\qquad \beta_{10}(x,t)u^{(1)}(x,t) + \beta_{01}(x,t)v^{(1)}(x,t) & \text{in } Q,\\
u^{(1)}(x,0) = u_0(x) \geq 0, \quad v^{(1)}(x,0) = v_0(x) \geq 0 & \text{in } \Omega,\\
u^{(1)}(x,t) = f_1(x,t) > 0, \quad v^{(1)}(x,t) = g(x,t) \geq 0 & \text{on } \Sigma.
\end{cases}
\tag{11}
$$

Then, $u^{(1)} \in C^{2+\alpha,1+\alpha/2}(\bar{Q})$ is the strictly positive solution of the heat equation,

First Order Linearisation – u

$$
\begin{cases}
\partial_t u^{(1)}(x,t) - \mu \Delta u^{(1)}(x,t) = 0 & \text{in } Q, \\
\partial_t v^{(1)}(x,t) - \nu \Delta v^{(1)}(x,t) = & \\
\qquad \beta_{10}(x,t) u^{(1)}(x,t) + \beta_{01}(x,t) v^{(1)}(x,t) & \text{in } Q, \\
u^{(1)}(x,0) = u_0(x) \geq 0, \quad v^{(1)}(x,0) = v_0(x) \geq 0 & \text{in } \Omega, \\
u^{(1)}(x,t) = f_1(x,t) > 0, \quad v^{(1)}(x,t) = g(x,t) \geq 0 & \text{on } \Sigma.
\end{cases}
\tag{11}
$$

Then, $u^{(1)} \in C^{2+\alpha, 1+\alpha/2}(\bar{Q})$ is the strictly positive solution of the heat equation, given by

$$
\int_0^t \int_\Omega \int_\Omega \Phi(x-y-z, t-s) \bar{f}^1(y,s) \bar{f}^2(z) \, dy \, dz \, ds + f_1(x,t) > 0,
$$

where Φ is the fundamental solution of the generalised heat equation

$$
\Phi(x,t) := (4\pi t)^{-n/2} e^{-\frac{\mu |x|^2}{4t}}.
$$

First Order Linearisation – v

Next, consider two different values of β_{10}, given by β_{10}^1 and β_{10}^2. Then for $j = 1, 2$, $v_j^{(1)}$ satisfies

$$\partial_t v_j^{(1)} - \nu \Delta v_j^{(1)} - \beta_{01}(x,t) v_j^{(1)} = \beta_{10}^j u^{(1)}. \qquad (12)$$

Therefore, $v_j^{(1)}$ is the unique solution given by

$$v_j^{(1)}(x,t) = \int_0^{T-t} \int_\Omega \Psi(x-y, T-t-s) \beta_{10}^j (y, T-s) u^{(1)}(y, T-s)\, dy\, ds,$$

where Ψ is the fixed, known Green's function for the operator $\partial_t - \nu\Delta - \beta_{01}$.

First Order Linearisation – v

Next, consider two different values of β_{10}, given by β_{10}^1 and β_{10}^2.
Then for $j = 1, 2$, $v_j^{(1)}$ satisfies

$$\partial_t v_j^{(1)} - \nu \Delta v_j^{(1)} - \beta_{01}(x,t) v_j^{(1)} = \beta_{10}^j u^{(1)}. \qquad (12)$$

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$$v_j^{(1)}(x,t) = \int_0^{T-t} \int_\Omega \Psi(x-y, T-t-s) \beta_{10}^j(y, T-s) u^{(1)}(y, T-s)\, dy\, ds,$$

where Ψ is the fixed, known Green's function for the operator $\partial_t - \nu\Delta - \beta_{01}$.
Note that $v_j^{(1)}$ is not yet determined.

First Order Linearisation – v

Next, consider two different values of β_{10}, given by β_{10}^1 and β_{10}^2.
Then for $j = 1, 2$, $v_j^{(1)}$ satisfies

$$\partial_t v_j^{(1)} - \nu \Delta v_j^{(1)} - \beta_{01}(x,t) v_j^{(1)} = \beta_{10}^j u^{(1)}. \qquad (12)$$

Therefore, $v_j^{(1)}$ is the unique solution given by

$$v_j^{(1)}(x,t) = \int_0^{T-t} \int_\Omega \Psi(x-y, T-t-s) \beta_{10}^j(y, T-s) u^{(1)}(y, T-s)\, dy\, ds,$$

where Ψ is the fixed, known Green's function for the operator
$\partial_t - \nu\Delta - \beta_{01}$.
Note that $v_j^{(1)}$ is not yet determined.
But, for $\beta_{10}^j(x,t) \geq 0$, $\beta_{01}(x,t) \leq 0$, we have that $v_j^{(1)} > 0$ since
$u^{(1)} > 0$.

Proof (Second Order Linearisation – $m = 1$)

Consider

$$u^{(2)} := \partial_\varepsilon^2 u\Big|_{\varepsilon=0}, \quad v^{(2)} := \partial_\varepsilon^2 v\Big|_{\varepsilon=0}.$$

Then, $(u^{(2)}, v^{(2)})$ solves

$$\begin{cases} \partial_t u^{(2)} - \mu\Delta u^{(2)} = 0 & \text{in } Q, \\ \partial_t v^{(2)} - \nu\Delta v^{(2)} = 2\beta_{20}[u^{(1)}]^2 + 2\beta_{11}u^{(1)}v^{(1)} & \\ \qquad\qquad\qquad + 2\beta_{02}[v^{(1)}]^2 + \beta_{10}u^{(2)} + \beta_{01}v^{(2)} & \text{in } Q, \\ u^{(2)}(x,0) = v^{(2)}(x,0) = 0 & \text{in } \Omega, \\ u^{(2)}(x,t) = f_2(x,t), \quad v^{(2)}(x,t) = 0 & \text{on } \Sigma. \end{cases}$$

Second Order Linearisation ($m = 1$) Main Theorem

Theorem

Assume that $F, G \in \mathcal{A}$ are such that $\beta_{11} = \beta_{02} \equiv 0$, and β_{01}, β_{20} are known and fixed, such that the compatibility and regularity assumptions are satisfied.
Let $\mathcal{M}^{+}_{G^j}$ be the measurement map associated to (2) for

$$u(x,t) = \sum_{l=1}^{2} \varepsilon^l f_l \quad \text{on } \Sigma.$$

If

$$\mathcal{M}^{+}_{G^1}(u|_{\partial\Omega}) = \mathcal{M}^{+}_{G^2}(u|_{\partial\Omega}),$$

then it holds that

$$\beta^1_{10}(x,t) = \beta^2_{10}(x,t) \text{ in } Q.$$

Second Order Linearisation ($m = 1$) Theorem Proof

$(u_j^{(2)}, v_j^{(2)})$ satisfy

$$\begin{cases} \partial_t u_j^{(2)} - \mu \Delta u_j^{(2)} = 0 & \text{in } Q, \\ \partial_t v_j^{(2)} - \nu \Delta v_j^{(2)} = 2\beta_{20}[u^{(1)}]^2 + \beta_{10} u_j^{(2)} + \beta_{01} v_j^{(2)} & \text{in } Q, \\ u_j^{(2)}(x,0) = v_j^{(2)}(x,0) = 0 & \text{in } \Omega, \\ u_j^{(2)}(x,t) = f_2^j(x,t), \quad v_j^{(2)}(x,t) = 0 & \text{on } \Sigma, \end{cases}$$

Second Order Linearisation ($m = 1$) Theorem Proof

$(u_j^{(2)}, v_j^{(2)})$ satisfy

$$\begin{cases} \partial_t u_j^{(2)} - \mu \Delta u_j^{(2)} = 0 & \text{in } Q, \\ \partial_t v_j^{(2)} - \nu \Delta v_j^{(2)} = 2\beta_{20}[u^{(1)}]^2 + \beta_{10} u_j^{(2)} + \beta_{01} v_j^{(2)} & \text{in } Q, \\ u_j^{(2)}(x,0) = v_j^{(2)}(x,0) = 0 & \text{in } \Omega, \\ u_j^{(2)}(x,t) = f_2^j(x,t), \quad v_j^{(2)}(x,t) = 0 & \text{on } \Sigma, \end{cases}$$

Then, $u_j^{(2)} \in C^{2+\alpha,1+\alpha/2}(\bar{Q})$ satisfies the heat equation for $j = 1, 2$.

Second Order Linearisation ($m = 1$) Theorem Proof

$(u_j^{(2)}, v_j^{(2)})$ satisfy

$$\begin{cases} \partial_t u_j^{(2)} - \mu \Delta u_j^{(2)} = 0 & \text{in } Q, \\ \partial_t v_j^{(2)} - \nu \Delta v_j^{(2)} = 2\beta_{20}[u^{(1)}]^2 + \beta_{10} u_j^{(2)} + \beta_{01} v_j^{(2)} & \text{in } Q, \\ u_j^{(2)}(x,0) = v_j^{(2)}(x,0) = 0 & \text{in } \Omega, \\ u_j^{(2)}(x,t) = f_2^j(x,t), \quad v_j^{(2)}(x,t) = 0 & \text{on } \Sigma, \end{cases}$$

Then, $u_j^{(2)} \in C^{2+\alpha,1+\alpha/2}(\bar{Q})$ satisfies the heat equation for $j = 1, 2$.
In this case, since $f_2^j(x,t)$ can be positive or negative, $u_j^{(2)}$ is not strictly positive,

Second Order Linearisation ($m = 1$) Theorem Proof

$(u_j^{(2)}, v_j^{(2)})$ satisfy

$$\begin{cases} \partial_t u_j^{(2)} - \mu\Delta u_j^{(2)} = 0 & \text{in } Q, \\ \partial_t v_j^{(2)} - \nu\Delta v_j^{(2)} = 2\beta_{20}[u^{(1)}]^2 + \beta_{10} u_j^{(2)} + \beta_{01} v_j^{(2)} & \text{in } Q, \\ u_j^{(2)}(x,0) = v_j^{(2)}(x,0) = 0 & \text{in } \Omega, \\ u_j^{(2)}(x,t) = f_2^j(x,t), \quad v_j^{(2)}(x,t) = 0 & \text{on } \Sigma, \end{cases}$$

Then, $u_j^{(2)} \in C^{2+\alpha,1+\alpha/2}(\bar{Q})$ satisfies the heat equation for $j = 1, 2$.
In this case, since $f_2^j(x,t)$ can be positive or negative, $u_j^{(2)}$ is not strictly positive, given by

$$u_j^{(2)}(x,t) = \int_0^t \int_\Omega \Phi(x-y, t-s) f_2^j(y,s)\, dy\, ds.$$

Second Order Linearisation ($m = 1$) Theorem Proof

$(u_j^{(2)}, v_j^{(2)})$ satisfy

$$\begin{cases} \partial_t u_j^{(2)} - \mu \Delta u_j^{(2)} = 0 & \text{in } Q, \\ \partial_t v_j^{(2)} - \nu \Delta v_j^{(2)} = 2\beta_{20}[u^{(1)}]^2 + \beta_{10} u_j^{(2)} + \beta_{01} v_j^{(2)} & \text{in } Q, \\ u_j^{(2)}(x,0) = v_j^{(2)}(x,0) = 0 & \text{in } \Omega, \\ u_j^{(2)}(x,t) = f_2^j(x,t), \quad v_j^{(2)}(x,t) = 0 & \text{on } \Sigma, \end{cases}$$

Then, $u_j^{(2)} \in C^{2+\alpha,1+\alpha/2}(\bar{Q})$ satisfies the heat equation for $j = 1, 2$.
In this case, since $f_2^j(x,t)$ can be positive or negative, $u_j^{(2)}$ is not strictly positive, given by

$$u_j^{(2)}(x,t) = \int_0^t \int_\Omega \Phi(x-y, t-s) f_2^j(y,s)\, dy\, ds.$$

When $\mathcal{M}^+_{G^1} = \mathcal{M}^+_{G^2}$, the input data satisfy $f_2^1 = f_2^2$, so $u_1^{(2)} = u_2^{(2)}$.

Second Order Linearisation ($m = 1$) Theorem Proof

Next, take the difference of the two equations for $j = 1, 2$:

$$\begin{aligned}\partial_t \tilde{v} - \nu \Delta \tilde{v} - \beta_{01} \tilde{v} &= \beta_{10} u_1^{(2)} - \beta_{10} u_2^{(2)} \\ &= \beta_{10}(u_1^{(2)} - u_2^{(2)}) + (\beta_{10} - \beta_{10}) u_2^{(2)}\end{aligned}$$

where $\tilde{v} = v_1^{(2)} - v_2^{(2)}$,

Second Order Linearisation ($m = 1$) Theorem Proof

Next, take the difference of the two equations for $j = 1, 2$:

$$\begin{aligned}\partial_t \tilde{v} - \nu \Delta \tilde{v} - \beta_{01} \tilde{v} &= \beta_{10} u_1^{(2)} - \beta_{10} u_2^{(2)} \\ &= \beta_{10}(u_1^{(2)} - u_2^{(2)}) + (\beta_{10} - \beta_{10}) u_2^{(2)}\end{aligned}$$

where $\tilde{v} = v_1^{(2)} - v_2^{(2)}$, since $u_1^{(2)} = u_2^{(2)}$ when $\mathcal{M}^+_{G^1} = \mathcal{M}^+_{G^2}$.

Second Order Linearisation ($m = 1$) Theorem Proof

Next, take the difference of the two equations for $j = 1, 2$:

$$\begin{aligned}\partial_t \tilde{v} - \nu \Delta \tilde{v} - \beta_{01} \tilde{v} &= \beta_{10} u_1^{(2)} - \beta_{10} u_2^{(2)} \\ &= \beta_{10}(u_1^{(2)} - u_2^{(2)}) + (\beta_{10} - \beta_{10}) u_2^{(2)} \\ &= (\beta_{10} - \beta_{10}) u_2^{(2)},\end{aligned}$$

where $\tilde{v} = v_1^{(2)} - v_2^{(2)}$, since $u_1^{(2)} = u_2^{(2)}$ when $\mathcal{M}^+_{G^1} = \mathcal{M}^+_{G^2}$.

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where $\tilde{v} = v_1^{(2)} - v_2^{(2)}$, since $u_1^{(2)} = u_2^{(2)}$ when $\mathcal{M}^+_{G^1} = \mathcal{M}^+_{G^2}$.
Therefore, $\tilde{v}$ is the unique solution given by

$$\tilde{v}(x,t) = \int_0^{T-t} \left[\Psi * ((\beta^1_{10} - \beta^2_{10})u_2^{(2)})\right](x, T-t-s)\, ds.$$

Second Order Linearisation ($m = 1$) Theorem Proof

Next, take the difference of the two equations for $j = 1, 2$:

$$\begin{aligned}\partial_t \tilde{v} - \nu \Delta \tilde{v} - \beta_{01} \tilde{v} &= \beta_{10} u_1^{(2)} - \beta_{10} u_2^{(2)} \\ &= \beta_{10}(u_1^{(2)} - u_2^{(2)}) + (\beta_{10} - \beta_{10}) u_2^{(2)} \\ &= (\beta_{10} - \beta_{10}) u_2^{(2)},\end{aligned}$$

where $\tilde{v} = v_1^{(2)} - v_2^{(2)}$, since $u_1^{(2)} = u_2^{(2)}$ when $\mathcal{M}^+_{G^1} = \mathcal{M}^+_{G^2}$.
Therefore, $\tilde{v}$ is the unique solution given by

$$\tilde{v}(x,t) = \int_0^{T-t} \left[\Psi * ((\beta_{10}^1 - \beta_{10}^2) u_2^{(2)}) \right] (x, T - t - s)\, ds.$$

satisfying the initial condition $v_j^{(1)}(x, 0) = 0$ for $j = 1, 2$.

Second Order Linearisation ($m = 1$) Theorem Proof

Next, take the difference of the two equations for $j = 1, 2$:

$$\begin{aligned} \partial_t \tilde{v} - \nu \Delta \tilde{v} - \beta_{01} \tilde{v} &= \beta_{10} u_1^{(2)} - \beta_{10} u_2^{(2)} \\ &= \beta_{10}(u_1^{(2)} - u_2^{(2)}) + (\beta_{10} - \beta_{10}) u_2^{(2)} \\ &= (\beta_{10} - \beta_{10}) u_2^{(2)}, \end{aligned}$$

where $\tilde{v} = v_1^{(2)} - v_2^{(2)}$, since $u_1^{(2)} = u_2^{(2)}$ when $\mathcal{M}^+_{G^1} = \mathcal{M}^+_{G^2}$. Therefore, the unique solution $\tilde{v}$ satisfies

$$\int_0^T \left[\Psi * ((\beta_{10}^1 - \beta_{10}^2) u_2^{(2)}) \right] (x, T - s) \, ds = 0.$$

satisfying the initial condition $v_j^{(1)}(x, 0) = 0$ for $j = 1, 2$.

Second Order Linearisation ($m = 1$) Theorem Proof

Next, take the difference of the two equations for $j = 1, 2$:

$$\begin{aligned}\partial_t \tilde{v} - \nu\Delta\tilde{v} - \beta_{01}\tilde{v} &= \beta_{10} u_1^{(2)} - \beta_{10} u_2^{(2)} \\ &= \beta_{10}(u_1^{(2)} - u_2^{(2)}) + (\beta_{10} - \beta_{10}) u_2^{(2)} \\ &= (\beta_{10} - \beta_{10}) u_2^{(2)},\end{aligned}$$

where $\tilde{v} = v_1^{(2)} - v_2^{(2)}$, since $u_1^{(2)} = u_2^{(2)}$ when $\mathcal{M}^+_{G^1} = \mathcal{M}^+_{G^2}$. Therefore, the unique solution $\tilde{v}$ satisfies

$$\int_0^T \left[\Psi * ((\beta^1_{10} - \beta^2_{10}) u_2^{(2)})\right](x, T - s)\, ds = 0.$$

This holds for all $u_2^{(2)}$, which depends on the input $f_2 \in C^{2+\alpha, 1+\alpha/2}(\bar{\Sigma})$.

Second Order Linearisation ($m = 1$) Theorem Proof

$$\int_0^T \left[\Psi * ((\beta_{10}^1 - \beta_{10}^2) u_2^{(2)})\right](x, T - s)\, ds = 0. \tag{13}$$

Second Order Linearisation ($m = 1$) Theorem Proof

$$\int_0^T \left[\Psi * ((\beta_{10}^1 - \beta_{10}^2) u_2^{(2)})\right] (x, T - s)\, ds = 0. \qquad (13)$$

Since G^j is continuous with respect to x and t, so is β_{10}^j, so there exists $\hat{\beta}_\eta(t)$ such that

$$\beta_{10}^1(x,t) - \beta_{10}^2(x,t) = \int_{-\infty}^{\infty} \hat{\beta}_\eta(t) e^{2\pi i \eta \cdot x}.$$

Second Order Linearisation ($m = 1$) Theorem Proof

$$\int_0^T \left[\Psi * ((\beta^1_{10} - \beta^2_{10})u_2^{(2)})\right](x, T-s)\, ds = 0. \tag{13}$$

Since G^j is continuous with respect to x and t, so is β^j_{10}, so there exists $\hat{\beta}_\eta(t)$ such that

$$\beta^1_{10}(x,t) - \beta^2_{10}(x,t) = \int_{-\infty}^{\infty} \hat{\beta}_\eta(t) e^{2\pi i \eta \cdot x}.$$

Choosing $u_2^{(2)}(x,t)$ to be the CGO solution $e^{-4\pi^2|\zeta|^2 t - \frac{2\pi i}{\sqrt{\mu}}\zeta\cdot x}$ which satisfies the second order linearised system, and is dense in the solution space.

Second Order Linearisation ($m = 1$) Theorem Proof

$$\int_0^T \left[\Psi * ((\beta_{10}^1 - \beta_{10}^2) u_2^{(2)})\right](x, T-s)\, ds = 0. \tag{13}$$

Since G^j is continuous with respect to x and t, so is β_{10}^j, so there exists $\hat{\beta}_\eta(t)$ such that

$$\beta_{10}^1(x,t) - \beta_{10}^2(x,t) = \int_{-\infty}^{\infty} \hat{\beta}_\eta(t) e^{2\pi i \eta \cdot x}.$$

Choosing $u_2^{(2)}(x,t)$ to be the CGO solution $e^{-4\pi^2|\zeta|^2 t - \frac{2\pi i}{\sqrt{\mu}}\zeta\cdot x}$ which satisfies the second order linearised system, and is dense in the solution space.
Applying the Fourier transform (in x) to (13),

Second Order Linearisation ($m = 1$) Theorem Proof

$$\int_0^T \left[\Psi * ((\beta_{10}^1 - \beta_{10}^2) u_2^{(2)})\right](x, T-s)\, ds = 0. \qquad (13)$$

Since G^j is continuous with respect to x and t, so is β_{10}^j, so there exists $\hat{\beta}_\eta(t)$ such that

$$\beta_{10}^1(x,t) - \beta_{10}^2(x,t) = \int_{-\infty}^{\infty} \hat{\beta}_\eta(t) e^{2\pi i \eta \cdot x}.$$

Choosing $u_2^{(2)}(x,t)$ to be the CGO solution $e^{-4\pi^2|\zeta|^2 t - \frac{2\pi i}{\sqrt{\mu}}\zeta\cdot x}$ which satisfies the second order linearised system, and is dense in the solution space.

Applying the Fourier transform (in x) to (13),

$\implies \hat{\beta}_\eta(t) = 0$ for all $\eta \in \mathbb{R}^n$,

Second Order Linearisation ($m = 1$) Theorem Proof

$$\int_0^T \left[\Psi * ((\beta_{10}^1 - \beta_{10}^2) u_2^{(2)})\right](x, T - s)\, ds = 0. \tag{13}$$

Since G^j is continuous with respect to x and t, so is β_{10}^j, so there exists $\hat{\beta}_\eta(t)$ such that

$$\beta_{10}^1(x,t) - \beta_{10}^2(x,t) = \int_{-\infty}^{\infty} \hat{\beta}_\eta(t) e^{2\pi i \eta \cdot x}.$$

Choosing $u_2^{(2)}(x,t)$ to be the CGO solution $e^{-4\pi^2|\zeta|^2 t - \frac{2\pi i}{\sqrt{\mu}}\zeta \cdot x}$ which satisfies the second order linearised system, and is dense in the solution space.

Applying the Fourier transform (in x) to (13),

$\implies \hat{\beta}_\eta(t) = 0$ for all $\eta \in \mathbb{R}^n$,

$\implies \beta_{10}^1(x,t) = \beta_{10}^2(x,t)$ for all $(x,t) \in Q$.

Extra Remark

Having determined β^j_{10}, we can now return to the first order linearisation to determine $v_j^{(1)}$:

$$v_j^{(1)}(x,t) = \int_0^{T-t} \int_\Omega \Psi(x-y, T-t-s) \beta^j_{10}(y, T-s) u^{(1)}(y, T-s)\, dy\, ds.$$

Extra Remark

Having determined β_{10}^{j}, we can now return to the first order linearisation to determine $v_j^{(1)}$:

$$v_j^{(1)}(x,t) = \int_0^{T-t}\int_\Omega \Psi(x-y, T-t-s)\beta_{10}^{j}(y, T-s)u^{(1)}(y, T-s)\,dy\,ds.$$

Furthermore, since $v^{(2)}$ satisfies

$$\begin{cases} \partial_t v^{(2)} - \nu\Delta v^{(2)} = 2\beta_{20}[u^{(1)}]^2 + 2\beta_{11}u^{(1)}v^{(1)} \\ \qquad\qquad\qquad\qquad +2\beta_{02}[v^{(1)}]^2 + \beta_{10}u^{(2)} + \beta_{01}v^{(2)} \qquad \text{in } Q, \\ v^{(2)}(x,0) = 0 \quad \text{in } \Omega, \quad v^{(2)}(x,t) = 0 \quad \text{on } \Sigma. \end{cases}$$

for β_{11}, β_{02} known, fixed and not necessarily equivalent to 0,

$$v_j^{(2)}(x,t) = \int_0^{T-t}\int_\Omega \Psi(x-y, T-t-s)[\mathbb{V}^{(2)}+\beta_{10}^{j}u_j^{(2)}](y, T-s)\,dy\,ds,$$

where $\mathbb{V}^{(2)}(x,t) := 2\beta_{20}[u^{(1)}]^2 + 2\beta_{11}u^{(1)}v^{(1)} + 2\beta_{02}[v^{(1)}]^2$.

Proof (Third Order Linearisation – $m = 2$)

Third order linearisation:

$$
\begin{cases}
\partial_t u^{(3)} - \mu \Delta u^{(3)} = 6\alpha_{30}[u^{(1)}]^3 + 6\alpha_{03}[v^{(1)}]^3 \\
\qquad\qquad\qquad +6\alpha_{12} u^{(1)}[v^{(1)}]^2 + 6\alpha_{21}[u^{(1)}]^2 v^{(1)} & \text{in } Q, \\
\partial_t v^{(3)} - \nu \Delta v^{(3)} = 6\beta_{30}[u^{(1)}]^3 + 6\beta_{03}[v^{(1)}]^3 \\
\qquad\qquad\qquad +6\beta_{12} u^{(1)}[v^{(1)}]^2 + 6\beta_{21}[u^{(1)}]^2 v^{(1)} \\
\qquad\qquad\qquad +6\beta_{20} u^{(1)} u^{(2)} + 6\beta_{02} v^{(1)} v^{(2)} \\
\qquad\qquad\qquad +3\beta_{11} u^{(2)} v^{(1)} + 3\beta_{11} u^{(1)} v^{(2)} \\
\qquad\qquad\qquad +\beta_{10} u^{(3)} + \beta_{01} v^{(3)} & \text{in } Q, \\
u^{(3)}(x,0) = v^{(3)}(x,0) = 0 & \text{in } \Omega, \\
u^{(3)}(x,t) = f_3(x,t), \quad v^{(3)}(x,t) = 0 & \text{on } \Sigma.
\end{cases}
\tag{14}
$$

Third Order Linearisation ($m = 2$) Main Theorem

Theorem

Assume that $F, G \in \mathcal{A}$ *are such that the compatibility and regularity assumptions are satisfied,*

$$\beta_{11} = \beta_{02} \equiv 0,$$

and all the remaining coefficients are known and fixed except for β_{20}*. Let* $\mathcal{M}^{+}_{F^j, G^j}$ *be the measurement map associated to* (2) *for*

$$u(x,t) = \sum_{l=1}^{3} \varepsilon^l f_l \quad \text{on } \Sigma.$$

If $\mathcal{M}^{+}_{F^1, G^1}(u|_{\partial\Omega}) = \mathcal{M}^{+}_{F^2, G^2}(u|_{\partial\Omega})$*, then*

$$\beta^1_{20}(x,t) = \beta^2_{20}(x,t) \quad \text{in } Q.$$

Third Order Linearisation ($m = 2$) Theorem Proof

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\begin{cases}
\partial_t u^{(3)} - \mu\Delta u^{(3)} = 6\alpha_{30}[u^{(1)}]^3 + 6\alpha_{03}[v^{(1)}]^3 \\
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Since all the coefficients of (14) are known and fixed except for β_{20}, we can compute $u^{(1)}$, $u_j^{(2)}$ and $v^{(1)}$ (given β_{10} known, fixed) using the first and second order linearised systems.

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Taking the difference for $j = 1, 2$, we have

$$\partial_t \tilde{v} - \nu \Delta \tilde{v} - \beta_{01} \tilde{v} = 6(\beta_{20}^1 - \beta_{20}^2) u^{(1)} u_2^{(2)}$$

when $\mathcal{M}^+_{F^1,G^1} = \mathcal{M}^+_{F^1,G^2}$.

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Applying the same argument with the same CGO solution for $u_2^{(2)}$,

$$(\beta_{20}^1 - \beta_{20}^2) u^{(1)} = 0 \quad \forall (x, t) \in Q.$$

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Fourth Order Linearisation $m = 3$

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\begin{cases}
\partial_t u^{(4)} - \mu \Delta u^{(4)} = 24\alpha_{40}[u^{(1)}]^4 + 24\alpha_{04}[v^{(1)}]^4 \\
\qquad +6\alpha_{31}[u^{(1)}]^3 v^{(1)} + 6\alpha_{13} u^{(1)}[v^{(1)}]^3 \\
\qquad +2\alpha_{22}[u^{(1)}]^2[v^{(1)}]^2 + 18\alpha_{30}[u^{(1)}]^2 u^{(2)} \\
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Obtain:

$$\alpha^1_{30}(x,t) = \alpha^2_{30}(x,t) \quad \text{in } Q.$$

Main Result (2) (General Case)

Theorem

Let $\mathcal{M}^{+}_{F^j,G^j}$ be the measurement map associated to (2) for $j = 1, 2$. Assume $F^j, G^j \in \mathcal{A}$ such that (3) holds. Suppose, for any

$$u(x,t) = \sum_{l=1}^{\infty} \varepsilon^l f_l \quad \text{on } \Sigma,$$

one has

$$\mathcal{M}^{+}_{F^1,G^1}(u|_{\partial\Omega}) = \mathcal{M}^{+}_{F^2,G^2}(u|_{\partial\Omega}).$$

2. $m \geq 2$: *if $\beta_{10}, \alpha_{m_2 n_2}$ are known, fixed for all $m_2 + n_2 = m + 1$, and*

$$\alpha_{m_1 n_1} \equiv 0 \quad \text{for all } 2 \leq m_1 + n_1 \leq m, m_1 \neq m,$$

then $$\alpha^1_{m0}(x,t) = \alpha^2_{m0}(x,t) \quad \text{in } Q.$$

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\qquad +6\beta_{02}[v^{(2)}]^2 + 6\beta_{11} u^{(2)} v^{(2)} \\
\qquad +3\beta_{11} u^{(3)} v^{(1)} + 3\beta_{11} u^{(1)} v^{(3)} \\
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\begin{cases}
\partial_t v^{(4)} - \nu \Delta v^{(4)} = 24\beta_{40}[u^{(1)}]^4 + 24\beta_{04}[v^{(1)}]^4 \\
\qquad +6\beta_{31}[u^{(1)}]^3 v^{(1)} + 6\beta_{13} u^{(1)}[v^{(1)}]^3 \\
\qquad +2\beta_{22}[u^{(1)}]^2[v^{(1)}]^2 + 18\beta_{30}[u^{(1)}]^2 u^{(2)} \\
\qquad +6\beta_{20} u^{(1)} u^{(3)} + 6\beta_{20}[u^{(2)}]^2 \\
\qquad +\beta_{10} u^{(4)} + \beta_{01} v^{(4)} + \beta_{00} & \text{in } Q, \\
v^{(4)}(x,0) = 0 \quad \text{in } \Omega, \qquad v^{(4)}(x,t) = 0 \quad \text{on } \Sigma.
\end{cases}
$$

Recall

$$
\begin{cases}
\partial_t u^{(3)} - \mu \Delta u^{(3)} = 6\alpha_{30}[u^{(1)}]^3 + 6\alpha_{03}[v^{(1)}]^3 \\
\qquad +6\alpha_{12} u^{(1)}[v^{(1)}]^2 + 6\alpha_{21}[u^{(1)}]^2 v^{(1)} & \text{in } Q, \\
u^{(3)}(x,0) = 0 \quad \text{in } \Omega, \qquad u^{(3)}(x,t) = f_3(x,t) \quad \text{on } \Sigma.
\end{cases}
$$

Fourth Order Linearisation $m = 3$

$$\begin{cases} \partial_t v^{(4)} - \nu\Delta v^{(4)} = 24\beta_{40}[u^{(1)}]^4 + 24\beta_{04}[v^{(1)}]^4 \\ \qquad +6\beta_{31}[u^{(1)}]^3 v^{(1)} + 6\beta_{13}u^{(1)}[v^{(1)}]^3 \\ \qquad +2\beta_{22}[u^{(1)}]^2[v^{(1)}]^2 + 18\beta_{30}[u^{(1)}]^2 u^{(2)} \\ \qquad +6\beta_{20}u^{(1)}u^{(3)} + 6\beta_{20}[u^{(2)}]^2 \\ \qquad +\beta_{10}u^{(4)} + \beta_{01}v^{(4)} + \beta_{00} & \text{in } Q, \\ v^{(4)}(x,0) = 0 \quad \text{in } \Omega, \qquad v^{(4)}(x,t) = 0 \quad \text{on } \Sigma. \end{cases}$$

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$$\begin{cases} \partial_t u^{(3)} - \mu\Delta u^{(3)} = 6\alpha_{30}[u^{(1)}]^3 + 6\alpha_{03}[v^{(1)}]^3 \\ \qquad +6\alpha_{12}u^{(1)}[v^{(1)}]^2 + 6\alpha_{21}[u^{(1)}]^2 v^{(1)} & \text{in } Q, \\ u^{(3)}(x,0) = 0 \quad \text{in } \Omega, \qquad u^{(3)}(x,t) = f_3(x,t) \quad \text{on } \Sigma. \end{cases}$$

But we do not manipulate the input f_3, but yes for f_2.

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\qquad + \\
\qquad +\beta_{10} u^{(4)} + \beta_{01} v^{(4)} + \beta_{00} & \text{in } Q, \\
v^{(4)}(x,0) = 0 \quad \text{in } \Omega, \qquad v^{(4)}(x,t) = 0 \quad \text{on } \Sigma.
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Recall

$$
\begin{aligned}
\partial_t u^{(4)} - \mu \Delta u^{(4)} &= 24\alpha_{40}[u^{(1)}]^4 + 24\alpha_{04}[v^{(1)}]^4 + 6\alpha_{31}[u^{(1)}]^3 v^{(1)} \\
&+ 6\alpha_{13} u^{(1)}[v^{(1)}]^3 + 2\alpha_{22}[u^{(1)}]^2[v^{(1)}]^2 + 18\alpha_{30}[u^{(1)}]^2 u^{(2)} \\
&+ 18\alpha_{03}[v^{(1)}]^2 v^{(2)} + 6\alpha_{12} u^{(2)}[v^{(1)}]^2 + 12\alpha_{12} u^{(1)} v^{(1)} v^{(2)} \\
&+ 12\alpha_{21} u^{(1)} u^{(2)} v^{(1)} + 6\alpha_{21}[u^{(1)}]^2 v^{(2)} \quad \text{in } Q
\end{aligned}
$$

Fourth Order Linearisation $m = 3$

$$\begin{cases} \partial_t v^{(4)} - \nu \Delta v^{(4)} = 24\beta_{40}[u^{(1)}]^4 + 24\beta_{04}[v^{(1)}]^4 \\ \qquad +6\beta_{31}[u^{(1)}]^3 v^{(1)} + 6\beta_{13} u^{(1)}[v^{(1)}]^3 \\ \qquad +2\beta_{22}[u^{(1)}]^2[v^{(1)}]^2 + 18\beta_{30}[u^{(1)}]^2 u^{(2)} \\ \qquad + \\ \qquad +\beta_{10} u^{(4)} + \beta_{01} v^{(4)} + \beta_{00} \qquad \text{in } Q, \\ v^{(4)}(x,0) = 0 \quad \text{in } \Omega, \qquad v^{(4)}(x,t) = 0 \quad \text{on } \Sigma. \end{cases}$$

Recall

$$\partial_t u^{(4)} - \mu \Delta u^{(4)} = 24\alpha_{40}[u^{(1)}]^4 + 24\alpha_{04}[v^{(1)}]^4 + 6\alpha_{31}[u^{(1)}]^3 v^{(1)} + 6\alpha_{13} u^{(1)}[v^{(1)}]^3 + 2\alpha_{22}[u^{(1)}]^2[v^{(1)}]^2 + 18\alpha_{30}[u^{(1)}]^2 u^{(2)} + 18\alpha_{03}[v^{(1)}]^2 v^{(2)} + 6\alpha_{12} u^{(2)}[v^{(1)}]^2 + 12\alpha_{12} u^{(1)} v^{(1)} v^{(2)} + 12\alpha_{21} u^{(1)} u^{(2)} v^{(1)} + 6\alpha_{21}[u^{(1)}]^2 v^{(2)} \quad \text{in } Q$$

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$\implies$ either $\beta_{10} \equiv 0$ or $\alpha_{30} = \alpha_{12} = \alpha_{21} = \alpha_{03} \equiv 0$.

Main Result (3) (General Case)

Theorem

Suppose, for any

$$u(x,t) = \sum_{l=1}^{\infty} \varepsilon^l f_l \quad \text{on } \Sigma,$$

one has

$$\mathcal{M}^+_{F^1,G^1}(u|_{\partial\Omega}) = \mathcal{M}^+_{F^2,G^2}(u|_{\partial\Omega}).$$

3. $m \geq 2$: *suppose* $\alpha_{mn}, \beta_{m_2,n_2}$ *are known, fixed for* $m_2 + n_2 = m+1$ *or* $m_2 + n_2 \leq 1$,

$$\beta_{m_1 n_1} \equiv 0 \quad \text{for all } 2 \leq m_1 + n_1 \leq m, m_1 \neq m,$$

If either $\alpha_{m_1 n_1} \equiv 0$ *for all* $2 \leq m_1 + n_1 \leq m$ *or* $\beta_{10} \equiv 0$,
then $\beta^1_{m0}(x,t) = \beta^2_{m0}(x,t)$ *in* Q.

Biological Applications: Reactive-Diffusive Predator-Prey Models

Our results can be applied to a variety of models. A group of examples is ecological differential systems with self diffusion given by diffusion constants $\mu, \nu > 0$:

$$
\begin{cases}
\partial_t u - \mu \Delta u = F(u, v) & \text{in } Q, \\
\partial_t v - \nu \Delta v = G(u, v) & \text{in } Q, \\
u, v \geq 0 & \text{in } Q, \\
u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 & \text{in } \Omega, \\
u = f \geq 0, \quad v = g \geq 0 & \text{on } \Sigma.
\end{cases}
\tag{15}
$$

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A Concrete Example - Cubic prey growth with hunting cooperation

$$\begin{cases} \partial_t u - \mu \Delta u = au^3 - (\lambda + \mu v)u^2 v & \text{in } Q, \\ \partial_t v - \nu \Delta v = bu - cv + (\alpha u - \beta v + \gamma uv)v + (\lambda + \mu v)u^2 v & \text{in } Q, \\ u, v \geq 0 & \text{in } Q, \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 & \text{in } \Omega, \\ u = f \geq 0, \quad v = g \geq 0 & \text{on } \Sigma. \end{cases}$$

Clearly, the origin $(0, 0)$ is an equilibrium point for this system.

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Clearly, the origin $(0,0)$ is an equilibrium point for this system.
Result 1: Suppose $\alpha(x,t) = \beta(x,t) \equiv 0$ and $c(x,t)$ known and fixed for each x, t. Let

$$u(x,t) = \sum_{l=1}^{2} \varepsilon^l f_l \quad \text{on } \Sigma.$$

Then,

$$b^1(x,t) = b^2(x,t) \text{ in } Q.$$

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$$\begin{cases} \partial_t u - \mu \Delta u = a u^3 - (\lambda + \mu v) u^2 v & \text{in } Q, \\ \partial_t v - \nu \Delta v = bu - cv + (\alpha u - \beta v + \gamma uv) v + (\lambda + \mu v) u^2 v & \text{in } Q, \\ u, v \geq 0 & \text{in } Q, \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 & \text{in } \Omega, \\ u = f \geq 0, \quad v = g \geq 0 & \text{on } \Sigma. \end{cases}$$

Clearly, the origin $(0,0)$ is an equilibrium point for this system.
Result 2: Suppose $\lambda(x,t) \equiv 0$. Let

$$u(x,t) = \sum_{l=1}^{4} \varepsilon^l f_l \quad \text{on } \Sigma.$$

Then,

$$a^1(x,t) = a^2(x,t) \quad \text{in } Q.$$

Concluding Remarks

- Our results can be easily extended to general second order parabolic operators of the form $\partial_t - \nabla \cdot (\sigma \nabla)$ for some fixed known measurable, bounded, coercive matrix $\sigma(x)$, by using the results of Caro-Kian, 2018.

Concluding Remarks

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- The restriction of positivity means that we can only derive the coefficient for u^m for $m \geq 1$. Although such assumptions are by no means restrictive, this still limits the types of physical models we can apply to, and the amount of information we can recover.

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- If positivity is not required, the boundary data of u and v can be used to fully determine the semilinear terms F and G, applying the results of Lin-Liu-Liu-Zhang, 2021.
- Yet, our measurement map only involves u, and no information is required for v.
- But, positivity is still crucial, for the results obtained to be physically realistic!

Publications

Thank you!

Hongyu Liu and Catharine W. K. Lo. *Determining a parabolic system by boundary observation of its non-negative solutions with applications*. In: *arXiv: 2303.13045* (2023).