Inverse source problems of local, nonlocal and nonlinear equations[1]

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[1]This talk is based on recent different joint works with Yavar Kian, Tony Liimatainen and Hongyu Liu.

Outline

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Inverse source problems for elliptic equations

Consider

$$\Delta u = F \text{ in } \Omega, \qquad u = f \text{ on } \partial\Omega. \tag{1.1}$$

An inverse source problem is to determine the source F by using the DN map. However, it can be seen that it is impossible to determine F due to the gauge invariance.

- **Gauge invariance.** Consider $\phi \in C_c^2(\Omega)$, and let $\widetilde{u} := u + \phi$, then $\Delta \widetilde{u} = F + \Delta\phi$. In general, $F + \Delta\phi \neq F$, but the Cauchy data of $\widetilde{u}$ and u are the same, i.e.,

$$\{\widetilde{u}|_{\partial\Omega}, \partial_\nu \widetilde{u}|_{\partial\Omega}\} = \{u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}\}.$$

Thus, it is natural to study similar questions for inverse source problems for nonlinear equations.

Inverse source problems for nonlinear elliptic equations

Theorem (Liimatainen-L., gauge invariance 2022)

Let $a_j(x,z) = a_j^{(1)}(x)z + a_j^{(2)}(x)z^2$*, where* $a_j^{(1)}, a_j^{(2)} \in C^\alpha(\overline{\Omega})$ *for some* $0 < \alpha < 1$*, for* $j = 1,2$*. Consider*

$$\begin{cases} \Delta u_j + a_j(x,u_j) = F_j & \text{in } \Omega, \\ u_j = f & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and let Λ_{a_j,F_j} *to be the corresponding DN map of* (1.2) *for* $j = 1,2$*. Suppose that*

$$\Lambda_{a_1,F_1}(f) = \Lambda_{a_2,F_2}(f) \text{ for any } f \in \mathscr{N}.$$

Then there exists $\psi \in C^{2,\alpha}(\overline{\Omega})$ *with* $\psi|_{\partial\Omega} = \partial_\nu \psi|_{\partial\Omega} = 0$ *in* Ω *such that*

$$\begin{cases} a_1^{(2)} = a_2^{(2)} =: a^{(2)}, \quad a_1^{(1)} = a_2^{(1)} + 2a^{(2)}\psi, \\ F_1 = F_2 - \Delta\psi - a_1^{(2)}\psi - a^{(2)}\psi^2. \end{cases} \tag{1.3}$$

In the nonlinear counterpart, we can further get uniqueness result for both coefficients and sources.

Corollary (Liimatainen-L., gauge breaking 2022)

For the quadratic case, assume additionally that

$$a_1^{(1)} = a_2^{(1)} \text{ in } \Omega$$

and

$$a_1^{(2)}(x) \neq 0 \text{ or } a_2^{(2)}(x) \neq 0 \text{ at any } x \in \Omega.$$

Then also

$$F_1 = F_2 \text{ and } a_1^{(2)} = a_2^{(2)} \text{ in } \Omega.$$

The proofs of both theorem and corollary are based on the higher order linearization.

The proof

1. Initiation.

We apply the higher order linearization method to the equation

$$\begin{cases} \Delta u_j + a_j^{(1)} u_j + a_j^{(2)} u_j^2 = F_j & \text{in } \Omega, \\ u_j = f_0 + \varepsilon_1 f_1 + \varepsilon_2 f_2 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

We denote $\varepsilon = (\varepsilon_1, \varepsilon_2)$, which especially means that $\varepsilon = 0$ is equivalent to $\varepsilon_1 = \varepsilon_2 = 0$. Below the index $j = 1, 2$ corresponds to the different sets of coefficients, and an index $\ell = 1, 2$ to ε_ℓ parameters. Let us denote by $u_j^{(0)}$ the solution to

$$\begin{cases} \Delta u_j^{(0)} + a_j^{(1)} u_j^{(0)} + a_j^{(2)} \left(u_j^{(0)}\right)^2 = F & \text{in } \Omega, \\ u_j^{(0)} = f_0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

□

2. First linearization.

Differentiate (1.4) with respect to ε_ℓ, for $\ell = 1, 2$. We obtain

$$\begin{cases} \left(\Delta + a_j^{(1)} + 2a_j^{(2)} u_j^{(0)}\right) v_j^{(\ell)} = 0 & \text{in } \Omega, \\ v_j^{(\ell)} = f_\ell & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where

$$v_j^{(\ell)} = \partial_{\varepsilon_\ell}\big|_{\varepsilon=0} u_j,$$

for $j, \ell = 1, 2$. The global uniqueness implies

$$Q := a_1^{(1)} + 2a_1^{(2)} u_1^{(0)} = a_2^{(1)} + 2a_2^{(2)} u_2^{(0)} \text{ in } \Omega. \tag{1.7}$$

It then follows by uniqueness of solutions to the Dirichlet problem (1.5) that

$$v^{(\ell)} := v_1^{(\ell)} = v_2^{(\ell)} \text{ in } \Omega, \quad \ell = 1, 2.$$

□

3. Second linearization.

For $j = 1, 2$, a straightforward computation shows that

$$\begin{cases} (\Delta + Q)\, w_j + 2a_j^{(2)} v^{(1)} v^{(2)} = 0 & \text{in } \Omega, \\ w_j = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

where $w_j = \partial^2_{\varepsilon_1 \varepsilon_2}\big|_{\varepsilon=0} u_j$. Let $\mathbb{V}^{(\ell)}$ $(\ell = 1, 2)$ be the solution of

$$\begin{cases} (\Delta + Q)\, \mathbb{V}^{(\ell)} = 0 & \text{in } \Omega, \\ \mathbb{V}^{(\ell)} = g_\ell & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

Multiply (1.8) by $\mathbb{V}^{(1)}$, then integration by parts gives rises to

$$\int_\Omega \left(a_1^{(2)} - a_2^{(2)} \right) v^{(1)} v^{(2)} \mathbb{V}^{(1)} \, dx = 0,$$

such that $\left(a_1^{(2)} - a_2^{(2)} \right) \mathbb{V}^{(1)} = 0$ in Ω. □

Semilinear reaction-diffusion equations

Consider the initial boundary value problem

$$\begin{cases} \rho(t,x)\partial_t u + \nabla\cdot(A(t,x)\nabla u) + b(t,x,u) = 0, & (t,x)\in(0,T)\times\Omega, \\ u(t,x) = f(t,x), & (t,x)\in(0,T)\times\partial\Omega, \\ u(0,x) = 0, & x\in\Omega. \end{cases}$$

Theorem (Kian-Liimatainen-L., 2023)

The same lateral DN map $\Lambda_{b_1}(f) = \Lambda_{b_2}(f)$, for all f, then there exists $\phi \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times\overline{\Omega})$ satisfying

$$\phi(0,x)=0, \quad x\in\Omega, \quad \phi(t,x) = \partial_{\nu_a}\phi(t,x) = 0, \quad (t,x)\in\Sigma.$$

such that $b_1 = S_\phi b_2$, where S_ϕ is defined by

$$S_\phi b(t,x,\mu) = b(t,x,\mu+\phi(t,x)) + \rho(t,x)\partial_t\phi(t,x) + \nabla\cdot(A\nabla\phi).$$

Outline

Fractional Laplacian

Let us start with some natural questions:

- What if the Laplacian $-\Delta$ is replaced by the fractional Laplacian $(-\Delta)^s$ for $0 < s < 1$? Can we consider the Calderón problem for the fractional Laplacian?

The fractional Laplacian is defined by

$$(-\Delta)^s u = \mathscr{F}^{-1}\left\{|\xi|^{2s}\widehat{u}(\xi)\right\}, \text{ for all } u \in \mathscr{S}(\mathbb{R}^n).$$

The fractional Laplacian $(-\Delta)^s$ is nonlocal: It does not preserve the supports, and computing $(-\Delta)^s u(x)$ involves values of u away from x.

The exterior value problem: Fractional Schrödinger equation

Since $(-\Delta)^s$ is a nonlocal operator, the forward problem for the fractional Schroödinger equation is given by (for all dimension $n \in \mathbb{N}$)

$$\begin{cases} (-\Delta)^s u + qu = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e := \mathbb{R}^n \setminus \Omega. \end{cases}$$

The well-posedness of the above equation can be guaranteed by the Lax-Milgram. Hence, one can derive that the DN map of the fractional Schrödinger equation will be given by

$$\Lambda_q : f \mapsto (-\Delta)^s u|_{\Omega_e},$$

where u is the unique solution to the fractional Schrödinger equation.

The answer of the fractional Calderón problem is positive.

Theorem (Ghosh-Rüland-Salo-Uhlmann, single measurement)

Let $q_j \in C^0(\overline{\Omega})$, Λ_{q_j} be the DN maps of $(-\Delta)^s u + q_j u = 0$ in Ω, and $W_j \subset \mathbb{R}^n \setminus \overline{\Omega}$ be arbitrary open set, for $j = 1, 2$. Then $\Lambda_{q_1}(f)|_{W_2} = \Lambda_{q_2}(f)|_{W_2}$ for one $0 \not\equiv f \in C_c^\infty(W_1)$. Then $q_1 = q_2$ in Ω.

Note that in the fractional case, the same DN map yields that

$$\int_\Omega (q_1 - q_2) u_1 u_2 \, dx = 0,$$

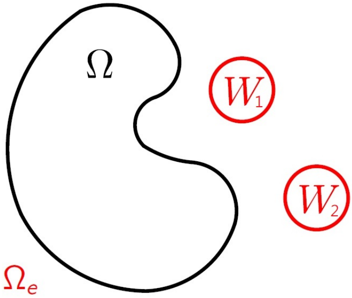
where u_1 and u_2 are the solutions in Ω with potentials q_1 and q_2.
The result can be shown by the unique continuation:

Theorem (Unique continuation)

Let $\mathscr{O} \subset \mathbb{R}^n$ be an arbitrary open set. If $(-\Delta)^s u = 0$ in $\mathscr{O}$ and $u = 0$ in a positive measurable subset of $\mathscr{O}$, one can conclude that $u \equiv 0$ in $\mathbb{R}^n$.

Main features

1. Partial data results for arbitrary open sets $W_1, W_2 \subset \Omega_e$ (W_1 and W_2 may not be disjoint)
2. The same method works for any dimensions $n \in \mathbb{N}$
3. New mechanism in solving nonlocal type inverse problems.



- Fractional Schrödinger equation (Ghosh-Salo-Uhlmann 2016).
- Nonlocal variable coefficients (Ghosh-L.Xiao 2017).
- Optimal stability (Rüland-Salo 2017).
- Nonlocal Schiffer (Cao-L.-Liu 2017).
- Monotonicity tests (Harrach-L. 2017, 2018).
- Single measurement and reconstruction (Ghosh-Rüland-Salo-Uhlmann 2018).
- Fractional conductivity (Covi 2018).
- Fractional Schrödinger equation with drift (Cékic-L.-Rüland 2018).
- Lipschitz stability with finite dimension (Rüland-Sincich 2018).
- Fractional semilinear (Lai-L. 2018, 2020).
- Fractional heat equation (Lai-L.-Rüland 2019).
- Directionally antilocal principal symbols (Covi-García-Ferrero-Rüland 2021).
- Fractional wave equation (Kow-L.-Wang 2021).
- Nonlocal elliptic operators (Ghosh-Uhlmann 2021).
- Fractional anisotropic on closed manifolds (Feizmohammadi-Ghosh-Krupchyk-Uhlmann 2021).
- Inverse source and minimal number of measurements (Liu-L. 2022).
- Global uniqueness of conductivity (Covi-Railo-Zimmermann 2022).
- Counterexample constructions with disjoint measured sets (Railo-Zimmermann 2022).
- Low regularity for γ (Railo-Zimmermann 2022).
- Nonlocal parabolic equation (Banerjee-Krishnan-Senapati 2022).
- Logarithmic stability (Covi-Railo-Tyni-Zimmermann 2022).
- Nonlocal parabolic operators (Lin-L.-Uhlmann 2022).
- Fractional elasticity (Covi-de Hoop-Salo 2022).
- Fractional p-Laplacian (Kar-L.-Zimmermann 2022).

Outline

Fractional elliptic equations

Consider the fractional equation

$$\begin{cases} (-\Delta)^s u(x) + \mathrm{a}(x,u) = F & \text{in } \Omega, \\ u = f & \text{in } \Omega_e, \end{cases} \tag{3.1}$$

where $0 < s < 1$. We are interested to determine

$$a(x,u) := \sum_{k=1}^{N} a^{(k)}(x)[u(x)]^k,$$

for $N \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, and F.

When $\mathrm{a}(x,u)$ is nonlinear, we need to assume the condition

$$a^{(1)}(x) = \partial_u \mathrm{a}(x,0) \geq 0 \text{ for } x \in \Omega,$$

in order to prove the local well-posedness of (3.1).

The case $s = 1$ and $a^{(k)} = 0$

Consider

$$\begin{cases} -\Delta u_j = F_j & \text{in } \Omega, \\ u_j = f & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

for $j = 1, 2$. In fact, to find the obstruction for the unique determination problem, let $\phi \in C_c^2(\Omega)$ be an arbitrary function, then one has $\phi = \partial_\nu \phi = 0$ on $\partial\Omega$. Let $(u_j|_{\partial\Omega}, \partial_\nu u_j|_{\partial\Omega})$ be the Cauchy data of (3.2), even if

$$(u_1|_{\partial\Omega}, \partial_\nu u_1|_{\partial\Omega}) = (u_2|_{\partial\Omega}, \partial_\nu u_2|_{\partial\Omega}),$$

but we can only prove the gauge invariance that $F_2 = F_1 - \Delta\phi$, and $\Delta\phi$ can be arbitrary. Therefore, the unique determination is not possible for the unknown sources in general.

Nonlocal Cauchy data

Let W_1 and W_2 be two arbitrary nonempty open subsets in Ω_e. It is always assumed that $\mathrm{supp}(f) \subset W_1$, and moreover $f|_{W_1} \in C^{2,s}(\overline{W_1})$. With the well-posedness at hand, we introduce the following exterior nonlocal partial Cauchy data set:

$$\begin{aligned} \mathscr{C}_{a,F}^{W_1,W_2}(f) &:= \left(u|_{W_1}, (-\Delta)^s u|_{W_2} \right) \\ &= \left(f|_{W_1}, (-\Delta)^s u|_{W_2} \right) \end{aligned} \tag{3.3}$$

where $u \in C^s(\mathbb{R}^n)$ is the unique solution to (3.1).

- The following results are holds at least when $\Omega \subset \mathbb{R}^n$ is a $C^{1,1}$ bounded domain and for any dimension $n \in \mathbb{N}$.

Minimal number of measurements

Theorem (L.-Liu, 2022)

Let $W_1, W_2 \subset \Omega_e$ be two arbitrary nonempty open subsets, and consider

$$\begin{cases} (-\Delta)^s u_j + \mathrm{a}_j(x, u_j) = F_j & \text{in } \Omega, \\ u_j = f & \text{in } \Omega_e, \end{cases} \tag{3.4}$$

where $\mathrm{a}_j(x, u_j) = \sum_{k=1}^{N} a_j^{(k)} [u_j(x)]^k$, $j = 1, 2$ and a finite $N \in \mathbb{N}$. Assuming the well-posedness of (3.4), if

$$\mathscr{C}_{\mathrm{a}_1, F_1}^{W_1, W_2}(f_k) = \mathscr{C}_{\mathrm{a}_2, F_2}^{W_1, W_2}(f_k), \ k = 0, 1, \ldots, N, \tag{3.5}$$

where $f_k \not\equiv f_l$, $0 \leq k, l \leq N$ and $k \neq l$, then one has

$$a_1^{(k)}(x) = a_2^{(k)}(x) \text{ in } \Omega, \ k = 1, 2, \ldots, N, \quad \text{and} \quad F_1 = F_2 \text{ in } \Omega.^{a}$$

[a] The number of unknowns equal to the number of measurements

The Proof

We want to show that

$$a_1^{(k)} = a_2^{(k)} \text{ in } \Omega, \text{ for } k = 1, 2, \ldots, N. \tag{3.6}$$

Let $f_0 = 0, f_1, \ldots, f_N \in Y_\delta$, which are mutually different, and consider $u_j^{(\ell)}$ to be the solutions of

$$\begin{cases} (-\Delta)^s u_j^{(\ell)} + \mathrm{a}_j(x, u_j^{(\ell)}) = F_j & \text{in } \Omega, \\ u_j^{(\ell)} = f_\ell & \text{in } \Omega_e, \end{cases} \tag{3.7}$$

for $\ell = 0, 1, \ldots, N$ and $j = 1, 2$.

By the strong uniqueness for the fractional Laplacian,

$$u^{(\ell)} := u_1^{(\ell)} = u_2^{(\ell)} \text{ in } \mathbb{R}^n, \quad \text{for } \ell = 0, 1, \ldots, N. \tag{3.8}$$

Moreover, via (3.7) and (3.8), it is not hard to derive

$$\sum_{k=1}^{N} a_j^{(k)} \left(u^{(\ell)} - u^{(0)} \right)^k = 0 \text{ in } \Omega,$$

for $j = 1, 2$, so that

$$\sum_{k=1}^{N} \left(a_1^{(k)} - a_2^{(k)} \right) \left(u^{(\ell)} - u^{(0)} \right)^k = 0 \text{ in } \Omega, \tag{3.9}$$

for all $\ell = 0, 1, \ldots, N$.

Rewrite (3.9) as $\mathrm{UA} = 0$ in Ω, where U is an $N \times N$ matrix

$$\mathrm{U} := \begin{pmatrix} u^{(1)} - u^{(0)} & \left(u^{(1)}\right)^2 - \left(u^{(0)}\right)^2 & \dots & \left(u^{(1)}\right)^N - \left(u^{(0)}\right)^N \\ u^{(2)} - u^{(0)} & \left(u^{(2)}\right)^2 - \left(u^{(0)}\right)^2 & \dots & \left(u^{(2)}\right)^N - \left(u^{(0)}\right)^N \\ \vdots & \vdots & \ddots & \vdots \\ u^{(N)} - u^{(0)} & \left(u^{(N)}\right)^2 - \left(u^{(0)}\right)^2 & \dots & \left(u^{(N)}\right)^N - \left(u^{(0)}\right)^N \end{pmatrix} \tag{3.10}$$

and A is an N-column vector

$$\mathrm{A} := \begin{pmatrix} a_1^{(1)} - a_2^{(1)} \\ a_1^{(2)} - a_2^{(2)} \\ \vdots \\ a_1^{(N)} - a_2^{(N)} \end{pmatrix}. \tag{3.11}$$

It suffices to show that the matrix U in (3.10) is non-singular a.e. in Ω.

Via direct computations, we have

$$\det \mathrm{U} = \det \begin{pmatrix} u^{(1)} - u^{(0)} & \left(u^{(1)}\right)^2 - \left(u^{(0)}\right)^2 & \dots & \left(u^{(1)}\right)^N - \left(u^{(0)}\right)^N \\ u^{(2)} - u^{(0)} & \left(u^{(2)}\right)^2 - \left(u^{(0)}\right)^2 & \dots & \left(u^{(2)}\right)^N - \left(u^{(0)}\right)^N \\ \vdots & \vdots & \ddots & \vdots \\ u^{(N)} - u^{(0)} & \left(u^{(N)}\right)^2 - \left(u^{(0)}\right)^2 & \dots & \left(u^{(N)}\right)^N - \left(u^{(0)}\right)^N \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & u^{(0)} & \left(u^{(0)}\right)^2 & \dots & \left(u^{(0)}\right)^N \\ 0 & u^{(1)} - u^{(0)} & \left(u^{(1)}\right)^2 - \left(u^{(0)}\right)^2 & \dots & \left(u^{(1)}\right)^N - \left(u^{(0)}\right)^N \\ 0 & u^{(2)} - u^{(0)} & \left(u^{(2)}\right)^2 - \left(u^{(0)}\right)^2 & \dots & \left(u^{(2)}\right)^N - \left(u^{(0)}\right)^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & u^{(N)} - u^{(0)} & \left(u^{(N)}\right)^2 - \left(u^{(0)}\right)^2 & \dots & \left(u^{(N)}\right)^N - \left(u^{(0)}\right)^N \end{pmatrix}$$

Hence,

$$\det \mathrm{U} = \det \begin{pmatrix} 1 & u^{(0)} & \left(u^{(0)}\right)^2 & \dots & \left(u^{(0)}\right)^N \\ 1 & u^{(1)} & \left(u^{(1)}\right)^2 & \dots & \left(u^{(1)}\right)^N \\ 1 & u^{(2)} & \left(u^{(2)}\right)^2 & \dots & \left(u^{(2)}\right)^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u^{(N)} & \left(u^{(N)}\right)^2 & \dots & \left(u^{(N)}\right)^N \end{pmatrix},$$

which is the Vandermonde matrix in the last identity and

$$\det \mathrm{U} = \prod_{1 \le \ell < m \le N} \left(u^{(m)} - u^{(\ell)}\right) \neq 0 \text{ a.e. in } \Omega.$$

Therefore, one can conclude that the vector A in (3.11) must be zero a.e. in Ω. Since each $a_j^{(k)} \in C^s(\overline{\Omega})$, for $j = 1, 2$, $k = 1, 2 \dots, N$, the claim (3.6) must hold. Finally, by using the equation (3.7), we can summarize that $F_1 = F_2$ in Ω as well.

Conclusions

- One can challenge open/unsolved inverse problems under fractional settings.
- Nonlocality is beneficial in solving related inverse problems.
- Some features of inverse problems are given:
  - **Local.** Recover coefficients then solutions.
  - **Nonlocal.** Recover solutions then coefficients.
    - ★ No linearization techniques are involved.
- As $s = 1$, $k = 2$, we[2] can only prove that there exists $\phi \in C_0^2(\Omega)$ such that there is a gauge invariance.
- On other hand, the uniqueness result for source holds for a nonlocal model[3].
- Gauge symmetry and breaking for parabolic equations are investigated.[4]
- Future study: Optimal condition to break the gauge.

[2] Liimatainen-L., 2022

[3] L.-Liu, 2022

[4] Kian-Liimatainen-L., in 2 weeks

Thank you for your attention !