Consistency of a Phase Field Regularisation for An Inverse Problem Governed by a Quasilinear Maxwell System

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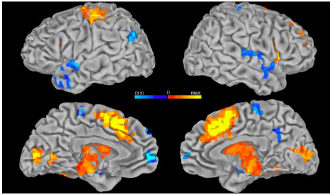
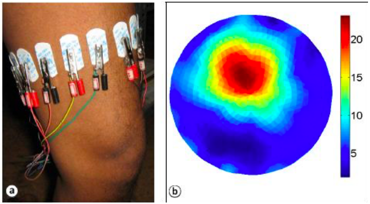
Motivation

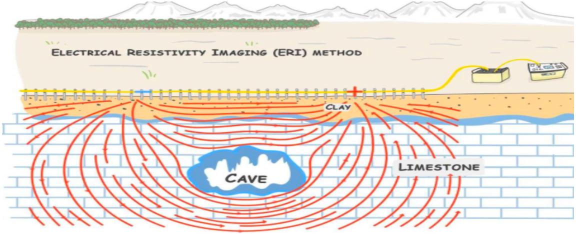
Electromagnetic non-destructive/non-invasive testing:



(a) fMRI



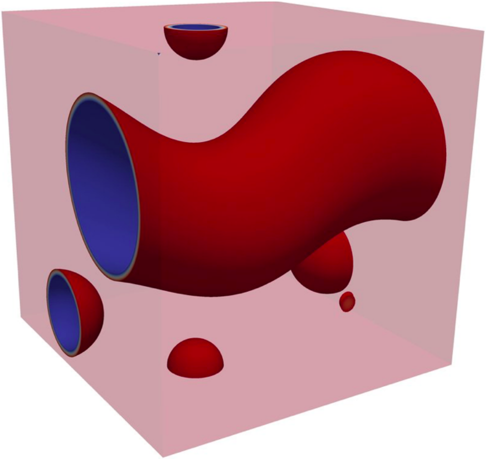
(b) Electrical Impedance Tomography



(c) Electrical Resistivity Tomography

The inverse problem

Identify the location of ferromagnetic materials (e.g. iron) in a mixture containing nonmagnetic materials (e.g. copper) from measurements.



Forward model derived from static Maxwell's equations in a medium.

The forward model

Magnetostatic equations in a medium:

$$\begin{aligned} \operatorname{div} \boldsymbol{B} &= 0 \qquad \text{Gauss's law for magnetism,} \\ \operatorname{curl} \boldsymbol{H} &= \boldsymbol{J} \qquad \text{Ampère law,} \end{aligned}$$

with magnetic induction $\boldsymbol{B}$ and magnetic field $\boldsymbol{H}$ related via $\mu \boldsymbol{H} = \boldsymbol{B}$.

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with magnetic induction $\boldsymbol{B}$ and magnetic field $\boldsymbol{H}$ related via $\mu \boldsymbol{H} = \boldsymbol{B}$.

Vector potential formulation

There exists unique vector potential $\boldsymbol{y}$ such that

$$\operatorname{curl} \boldsymbol{y} = \boldsymbol{B}, \quad \operatorname{div} \boldsymbol{y} = 0,$$

leading to the forward model

$$\begin{cases} \operatorname{curl}\left(\mu^{-1} \operatorname{curl} \boldsymbol{y}\right) = \boldsymbol{J} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{y} = 0 & \text{in } \Omega, \\ \boldsymbol{y} \times \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

with perfectly conducting electric boundary conditions.

The B-H curve

Constitutive relation:

$$\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B} =: \nu\boldsymbol{B},$$

with magnetic permeability μ (or magnetic reluctivity $\nu = \frac{1}{\mu}$).

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with magnetic permeability μ (or magnetic reluctivity $\nu = \frac{1}{\mu}$).

- $\nu = \nu_0$ is constant for non-magnetic materials.
- For ferromagnetic materials, ν may depend nonlinearly on $|\boldsymbol{B}|$, i.e., $\boldsymbol{H} = f(\boldsymbol{B})$ where $f(\boldsymbol{s}) = \nu(|\boldsymbol{s}|)\boldsymbol{s}$.

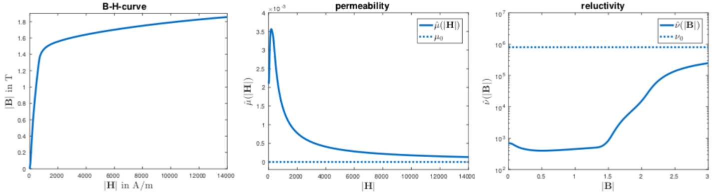

Figure: Left: B-H curve $\frac{1}{f}$ of a ferromagnetic material. Center: Magnetic permeability μ. Right: Magnetic reluctivity ν on log scale. from Ph.D. thesis of P. Gangl

- Magnetic hysteresis is neglected here.

Forward model

For $u \in L^1(\Omega;[0,1]) := \{g \in L^1(\Omega) \,:\, 0 \le g \le 1\}$ define interpolation reluctivity

$$\nu(u,\boldsymbol{y}) = \nu_0(1-u) + \nu_1(|\mathrm{curl}\,\boldsymbol{y}|)u,$$

so that for $u = \chi_{\Omega_1}$,

$$\nu = \begin{cases} \nu_0 & \text{in } \Omega_0 = \Omega \setminus \overline{\Omega_1} \text{ (nonmagnetic region),} \\ \nu_1(|\boldsymbol{B}|) & \text{in } \Omega_1 \text{ (magnetic region).} \end{cases}$$

Hence,

knowing u $\quad\Leftrightarrow\quad$ knowing the location of Ω_1 and Ω_0.

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Forward model

$$\begin{cases} \mathrm{curl}\left([\nu_0(1-u) + u\nu_1(|\mathrm{curl}\, \boldsymbol{y}|)] \mathrm{curl}\, \boldsymbol{y} \right) = \boldsymbol{J} & \text{in } \Omega, \\ \mathrm{div}\, \boldsymbol{y} = 0 & \text{in } \Omega, \\ \boldsymbol{y} \times \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

A quasilinear curl-curl system with divergence-free constraint, and the implicit constraint $\mathrm{div} \boldsymbol{J} = 0$!

Properties

Let $\Omega \subset \mathbb{R}^3$ be Lipschitz polyhedral, simply connected, and $\nu_1 \in C^0(\mathbb{R})$.
Assume

- $\exists$ constants $\underline{\nu} \in (0, \nu_0), \overline{\nu} \in [\nu_0, \infty)$ such that $\underline{\nu} \leq \nu_1(s) \leq \nu_0$ and

$$(\nu_1(s)s - \nu_1(r)r)(s - r) \geq \underline{\nu}|s - r|^2 \quad \text{(strong monotoncity)},$$
$$|\nu_1(s)s - \nu_1(r)r| \leq \overline{\nu}|s - r| \quad \text{(Lipschitz continuity)}.$$

- $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ and $u \in L^1(\Omega; [0, 1])$.

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Via a nonlinear saddle point formulation, for $\boldsymbol{Z} = H_0(\mathrm{curl}) \cap H(\mathrm{div} = 0)$

- (Well-posedness) $\exists$! weak solution pair $(\boldsymbol{y}, \phi) \in \boldsymbol{Z} \times H_0^1(\Omega)$.
- (Continuity) If $u_k \to u$ in $L^1(\Omega)$, then

$$\boldsymbol{y}_k \to \boldsymbol{y} \text{ strongly in } \boldsymbol{Z}, \quad \phi_k \to \phi \text{ weakly in } H_0^1(\Omega).$$

- Induces a solution mapping $\boldsymbol{S} : u \mapsto \boldsymbol{y}(u)$.

Inverse problem

Inverse problem:

$$(I) \quad \text{find } u \in L^1(\Omega, \{0,1\}) \text{ s.t. } \boldsymbol{G} \circ \boldsymbol{S}(u) = \boldsymbol{y}_m \text{ in } \mathcal{O}$$

where

- $\boldsymbol{y}_m$ is a measurement;
- $\mathcal{O}$ is a Hilbert space;
- $\boldsymbol{G} : \boldsymbol{Z} \to \mathcal{O}$ Lipschitz continuous and bounded observation operator.

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Examples:

- Ω Lipschitz polyhedral, $\mathcal{O} = \boldsymbol{L}^2(D)$ for subdomain $D \subset \Omega$, $\boldsymbol{G}(\boldsymbol{y}) = \boldsymbol{y}|_D$ (Interior measurements).
- Ω convex polyhedral/of class $C^{1,1}$, $\mathcal{O} = \boldsymbol{L}^2(\Sigma)$ for $\Sigma \subset \partial\Omega$, $\boldsymbol{G}(\boldsymbol{y}) = \boldsymbol{y}|_\Sigma$ (Boundary measurements).

Likely that (I) is ill-posedness $\therefore$ regularization is needed!

Perimeter/Total variation regularization

Overcome illposedness of (I) with

$$(I^\alpha) \quad \text{find } u^\alpha = \underset{v \in BV(\Omega;\{0,1\})}{\arg\min} \left(\alpha TV(v) + \frac{1}{2} \|\boldsymbol{G} \circ \boldsymbol{S}(v) - \boldsymbol{y}_m\|_{\mathcal{O}}^2 \right),$$

where $TV(v) = \sup\{\int_\Omega v \, \mathrm{div}\phi \text{ s.t. } \phi \in C_0^1(\Omega;\mathbb{R}^3), \|\phi\|_\infty \leq 1\}$.

This is perimeter regularization, i.e., the boundary $\partial\{u^\alpha = 1\}$ should have finite perimeter.

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Standard analysis yields

- (Existence) For any $\alpha > 0$, $\exists u^\alpha \in BV(\Omega, \{0,1\})$ to (I^α).
- (Continuity) If $\boldsymbol{y}_m^n \to \boldsymbol{y}_m$ in $\mathcal{O}$, and u_n^α solves (I^α) with data $\boldsymbol{y}_m^n$. Then,

  $$u_n^\alpha \to u^\alpha \text{ in } L^1(\Omega), \quad TV(u_n^\alpha) \to TV(u^\alpha),$$

  with u^α solves (I^α) with data $\boldsymbol{y}_m$.

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  $$u_n^\alpha \to u^\alpha \text{ in } L^1(\Omega), \quad TV(u_n^\alpha) \to TV(u^\alpha),$$

  with u^α solves (I^α) with data $\boldsymbol{y}_m$.
- (Consistency) If (I) has a solution $u^* \in BV(\Omega;\{0,1\})$, and u_δ^α solves (I^α) with data $\boldsymbol{y}_m^\delta$ such that $\|\boldsymbol{y}_m^\delta - \boldsymbol{y}_m\|_{\mathcal{O}} \leq \delta$. Then, choosing $(\alpha_\delta)_{\delta>0}$ such that $\delta^2/\alpha_\delta \to 0$, it holds that

  $$u^{\alpha_\delta} \to w \text{ in } L^1(\Omega)$$

  and w is a minimum-variation solution to (I).

Phase field regularization

Non-convexity of $BV(\Omega, \{0,1\})$ is difficult for numerical implementation. Thus, approximate $TV(\cdot)$ by the Ginzburg–Landau functional

$$E_\varepsilon(v^\varepsilon) = \frac{8}{\pi} \int_\Omega \frac{\varepsilon}{2} |\nabla v^\varepsilon|^2 + \frac{1}{\varepsilon} v^\varepsilon (1 - v^\varepsilon).$$

Well-known result of Modica (1987) shows $E_\varepsilon(\cdot) \xrightarrow{\Gamma} TV(\cdot)$ as $\varepsilon \to 0$.

Formally: as $\varepsilon \to 0$, $v^\varepsilon \to v \in BV(\Omega, \{0,1\})$ in suitable sense.

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Benefits?

- Change solution space from (non-convex) $BV(\Omega, \{0,1\})$ to a convex space $\mathcal{K} := \{f \in H^1(\Omega) : 0 \leq f(x) \leq 1 \text{ a.e. in } \Omega\}$.
- Easier to devise numerical algorithms involving first order optimality conditions.

Properties of the Phase field inverse problem

$$(I_\varepsilon^\alpha) \quad \text{find } u_\varepsilon^\alpha = \underset{v\in\mathcal{K}}{\arg\min}\left(\alpha E_\varepsilon(v) + \frac{1}{2}\|\boldsymbol{G}\circ\boldsymbol{S}(v) - \boldsymbol{y}_m\|_{\mathcal{O}}^2\right),$$

where $\mathcal{K} = \{f \in H^1(\Omega) : 0 \le f \le 1 \text{ a.e. in } \Omega\}$.

Note: $E_\varepsilon(v) = \frac{8}{\pi}\int_\Omega \frac{\varepsilon}{2}|\nabla v|^2 + \frac{1}{\varepsilon}v(1-v)$ is nonnegative over the set $\mathcal{K}$!

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Properties

- (Existence) For $\alpha, \varepsilon > 0$, $\exists u_\varepsilon^\alpha \in \mathcal{K}$ to (I_ε^α).
- (Continuity) If $\boldsymbol{y}_m^n \to \boldsymbol{y}_m$ in $\mathcal{O}$, and $u_{\varepsilon,n}^\alpha \in \mathcal{K}$ solves (I_ε^α) with data $\boldsymbol{y}_m^n$. Then,

$$u_{\varepsilon,n}^\alpha \to u_\varepsilon^\alpha \text{ in } H^1(\Omega),$$

with u_ε^α a solution to (I_ε^α) with data $\boldsymbol{y}_m$.

Consistency as $\varepsilon \to 0$

Behavior as $\varepsilon \to 0$

For fixed $\alpha, \varepsilon > 0$, let $u_\varepsilon^\alpha \in \mathcal{K}$ be a solution to (I_ε^α). Then, there exists a solution $u_*^\alpha \in BV(\Omega, \{0,1\})$ to (I^α) such that

$$u_\varepsilon^\alpha \to u_*^\alpha \text{ in } L^1(\Omega), \quad J_\varepsilon(u_\varepsilon^\alpha) \to J(u_*^\alpha) \quad \text{as } \varepsilon \to 0.$$

- Classical result using Gamma convergence $E_\varepsilon(\cdot) \xrightarrow{\Gamma} TV(\cdot)$.
- $J_f(u) = \frac{1}{2}\|\boldsymbol{G} \circ \boldsymbol{S}(u) - \boldsymbol{y}_m\|_{\mathcal{O}}^2$ is a continuous perturbation.
- This shows consistency of the phase field regularisation and validates its use.

Consistency as $\varepsilon, \alpha \to 0$

New result

If (I) has a solution $u_* \in BV(\Omega, \{0,1\})$ with $\partial\{u_* = 1\}$ smooth and $\mathrm{curl}\, \boldsymbol{S}(u_*) \in \boldsymbol{L}^{2+}(\Omega)$. For any $(\varepsilon_k)_{k\in\mathbb{N}} \to 0$, choose $(\alpha_k)_{k\in\mathbb{N}} \to 0$ s.t.

$$\limsup_{k\to\infty} \frac{\varepsilon_k^2}{\alpha_k} = 0, \qquad (\star)$$

then there exists a solution $u \in BV(\Omega, \{0,1\})$ to (I) such that

$$u_{\varepsilon_k}^{\alpha_k} \to u \text{ in } L^1(\Omega), \quad TV(u) \leq TV(u_*).$$

Compare to the Consistency of TV solutions

If (I) has a solution $u^* \in BV(\Omega; \{0,1\})$, and u_δ^α solves (I^α) with data $\boldsymbol{y}_m^\delta$ s.t. $\|\boldsymbol{y}_m^\delta - \boldsymbol{y}_m\|_{\mathcal{O}} \leq \delta$. Then, choosing $(\alpha_\delta)_{\delta>0}$ s.t. $\delta^2/\alpha_\delta \to 0$, it holds

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Remark:

- Smoothness assumption "with ... $\boldsymbol{L}^{2+}(\Omega)$" can be dropped, but the relation $(\star)$ is replaced with something less explicit:

$$\limsup_{\varepsilon_k\to 0} \frac{1}{\alpha_k} \|(w_{\varepsilon_k} - u_*)\mathrm{curl}\boldsymbol{S}(u_*)\|^2_{\boldsymbol{L}^2(\Omega)} = 0,$$

  where $w_{\varepsilon_k} \to u_*$ in $L^1(\Omega)$ as $k \to \infty$ is a recovery sequence in Gamma convergence.

Optimality conditions - variational inequality

Let $\boldsymbol{G} : \boldsymbol{Z} \to \mathcal{O}$ be continuously Fréchet differentiable, and let $u_\varepsilon^\alpha \in \mathcal{K}$ be a solution to (I_ε^α). Then,

$$\int_\Omega \Big((\nu_0 - \nu_1(|\mathrm{curl}\, \boldsymbol{y}_\varepsilon^\alpha|)) \mathrm{curl}\, \boldsymbol{y}_\varepsilon^\alpha \cdot \mathrm{curl}\, \boldsymbol{q}_\varepsilon^\alpha + \frac{\alpha 8}{\pi \varepsilon}(1 - 2u_\varepsilon^\alpha) \Big)(w - u_\varepsilon^\alpha) + \int_\Omega \alpha \frac{8}{\pi} \nabla u_\varepsilon^\alpha \cdot \nabla (w - u_\varepsilon^\alpha) \geq 0 \quad \forall w \in \mathcal{K}, \qquad (\dagger)$$

where the adjoint $\boldsymbol{q}_\varepsilon^\alpha$ satisfies a linear saddle point problem.

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Does optimality condition $(\dagger)$ converge as $\varepsilon \to 0$?

Problems:

- Q1 What is the optimality condition for (I^α)?
- Q2 Can we pass to the limit $\varepsilon \to 0$ rigorous?

Limit as $\varepsilon \to 0$?

A1 - Express (I^{α}) as a shape optimization problem, and derive the shape gradient.

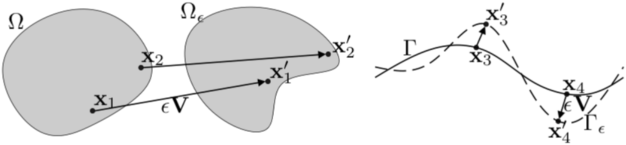

Figure: Perturb Ω by suitable velocity fields V and compute the change of the solution $\boldsymbol{y}(\Omega)$ with respect to V. Figure taken from book by S. Walker.

A2 - Derive a related optimality conditions for (I_ε^{α}) using domain variation, and then pass to the limit $\varepsilon \to 0$.

Some details...

Admissible domain variation

Velocity field $V \in C^0([0,\tau]; C_c^2(\Omega;\mathbb{R}^3))$ induces transformation $T : [0,\tau] \times \Omega \to \Omega$ with $T_t(x) = T(t,x)$, $T(0,x) = x$.

If u^α solves (I^α) with $\partial\{u^\alpha = 1\}$ Lipschitz then

$$u^\alpha \circ T_t^{-1} \in BV(\Omega, \{0,1\})$$

and

$$J(u^\alpha) \leq J(u^\alpha \circ T_t^{-1}) \quad \Rightarrow \quad \partial_t J(u^\alpha \circ T_t^{-1})|_{t=0} =: \mathrm{D}J(u^\alpha)[V] = 0.$$

From shape calculus

- $\dot{\boldsymbol{y}}^\alpha[V] = \partial_t \boldsymbol{S}(u^\alpha \circ T_t^{-1})|_{t=0}$ (shape derivative) satisfies a linear saddle point problem;
- $DJ(u^\alpha)[V] = \partial_t J(u^\alpha \circ T_t^{-1})|_{t=0}$ (shape gradient) yields the optimality condition $DJ(u^\alpha)[V] = 0$.

Domain variation optimality condition

Similarly, for the PF inverse problem, if u_ε^α is a solution to (I_ε^α), then

$$J_\varepsilon(u_\varepsilon^\alpha) \leq J_\varepsilon(u_\varepsilon^\alpha \circ T_t^{-1}) \quad \Rightarrow \quad \partial_t J_\varepsilon(u_\varepsilon^\alpha \circ T_t^{-1})|_{t=0} =: \mathrm{D}J_\varepsilon(u_\varepsilon^\alpha)[V] = 0.$$

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Example - $C^{1,1}$-boundary and $\mathcal{O} = \boldsymbol{L}^2(\partial\Omega)$ (boundary measurement):

$$DJ_\varepsilon(u_\varepsilon^\alpha)[V] = \int_{\partial\Omega} (\boldsymbol{y}_\varepsilon^\alpha - \boldsymbol{y}_m) \cdot \dot{\boldsymbol{y}}_\varepsilon^\alpha[V] + \int_\Omega \frac{8\varepsilon}{\pi} \nabla u_\varepsilon^\alpha \cdot (\nabla V) \nabla u_\varepsilon^\alpha$$
$$+ \frac{8}{\pi} \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon^\alpha|^2 + \frac{1}{\varepsilon} u_\varepsilon^\alpha (1 - u_\varepsilon^\alpha) \right) \mathrm{div} V$$

With more regularity, can be shown to be equivalent to the variational inequality (†)!

Convergence of optimality conditions

Problems:

Q1 What is the optimality condition for (I^α)? ✓

Q2 Can we pass to the limit $\varepsilon \to 0$ rigorous? ✓

Theorem: All the important things converge

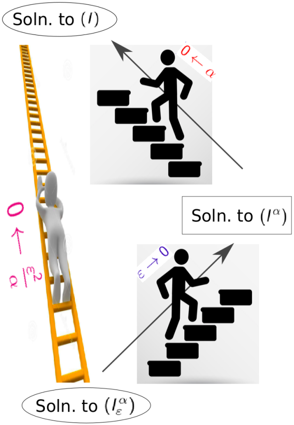
Fix $\alpha > 0$, then

$$u_\varepsilon^\alpha \to u^\alpha \text{ in } L^1(\Omega), \quad J_\varepsilon(u_\varepsilon^\alpha) \to J(u^\alpha) \text{ in } \mathbb{R},$$

and for any $V \in C^0([0,\tau]; C_c^2(\Omega;\mathbb{R}^3))$, it holds that

$$\text{(shape derivative) } \dot{\boldsymbol{y}}_\varepsilon^\alpha[V] \rightharpoonup \dot{\boldsymbol{y}}^\alpha[V] \text{ in } \boldsymbol{H}^1(\Omega),$$
$$\text{(optimality condition) } \mathrm{D}J_\varepsilon(u_\varepsilon^\alpha)[V] \to \mathrm{D}J(u^\alpha)[V] \text{ in } \mathbb{R}.$$

Summary



Thank you for your attention!