COLUMBIA | ENGINEERING
The Fu Foundation School of Engineering and Applied Science

Discovering Low-Dimensional State Variables from High-Dimensional Observation Data: A Means to Model Complex Dynamical Systems

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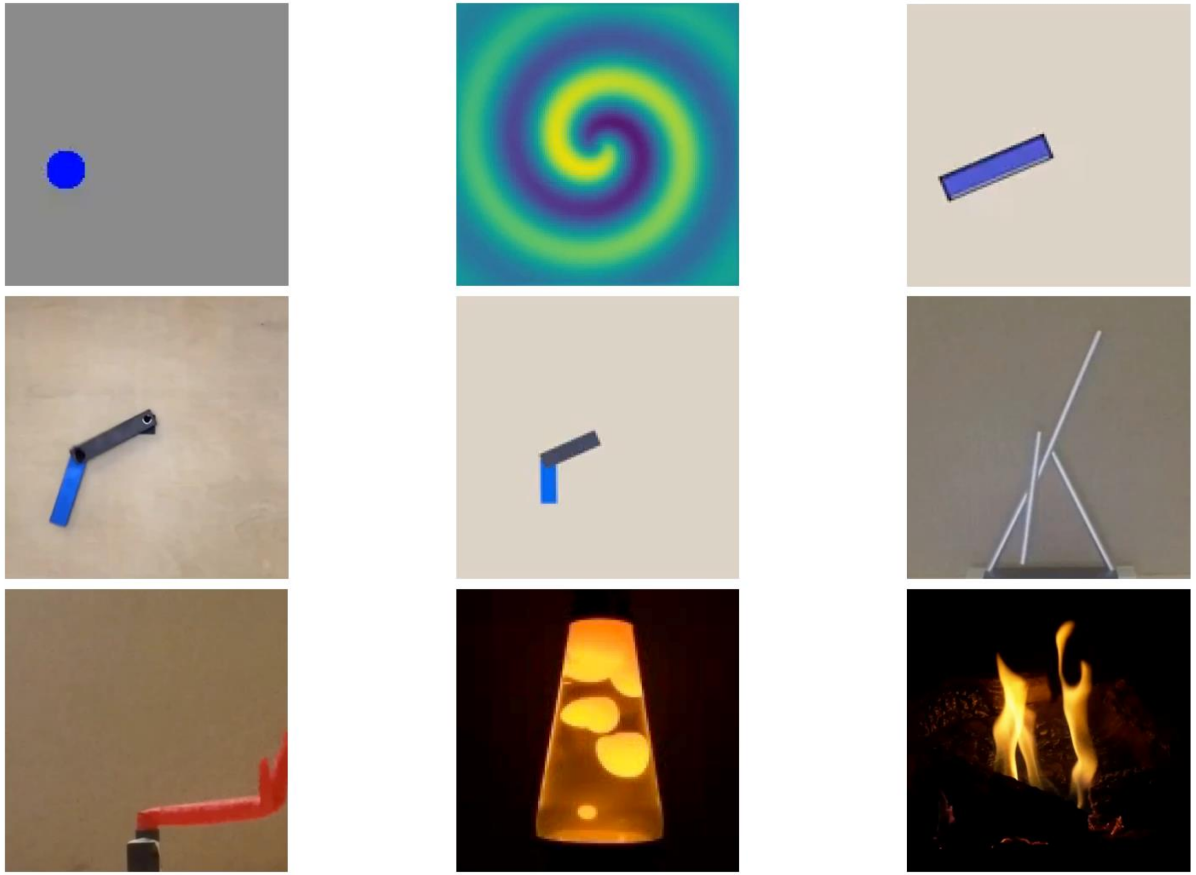
Joint work with Boyuan Chen, Ishaan Chandratreya, Sunand Raghupathi, Qiang Du, and Hod Lipson.

Overview

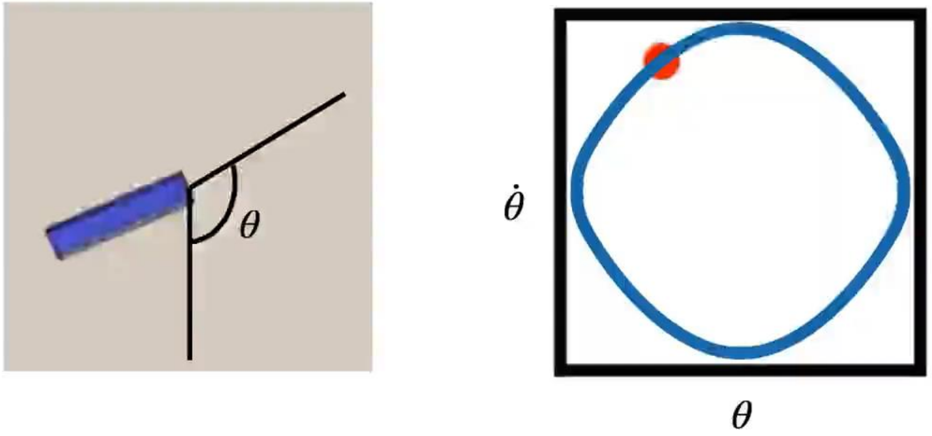
- Modelling dynamical systems from experimental data
- Low-dimensional state variables: how to discover?
- Long-term dynamics prediction: how to improve?
- Summary and underlying mathematics

Modelling dynamical systems from experimental data

Dynamical systems with visual observations

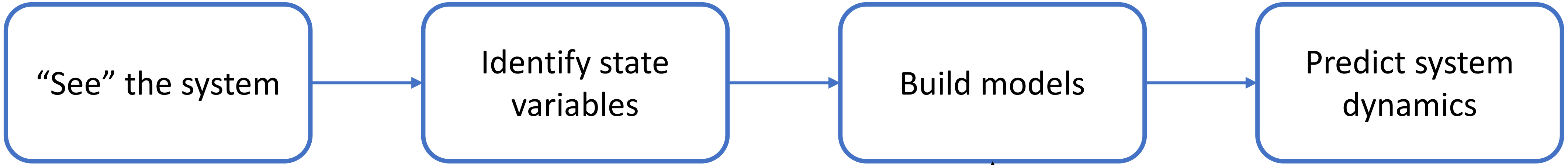


Modelling dynamical systems



$$X = [\theta, \omega] \qquad \begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = -\dfrac{3g}{2L}\sin\theta \end{cases}$$

Modelling dynamical systems



"See" the system

Identify state variables

Build models

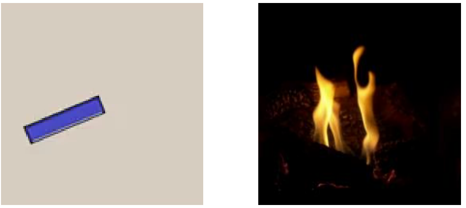
Predict system dynamics

High-dimensional visual observations (e.g., D=128*128*3=49152).
Redundant pixel information.

State variables: **complete and non-redundant** set of variables describing system dynamics.
Low-dimensional.

Learn models with state variables[1,2]

Direct measurement of state variables:
Expensive, Require domain knowledge.



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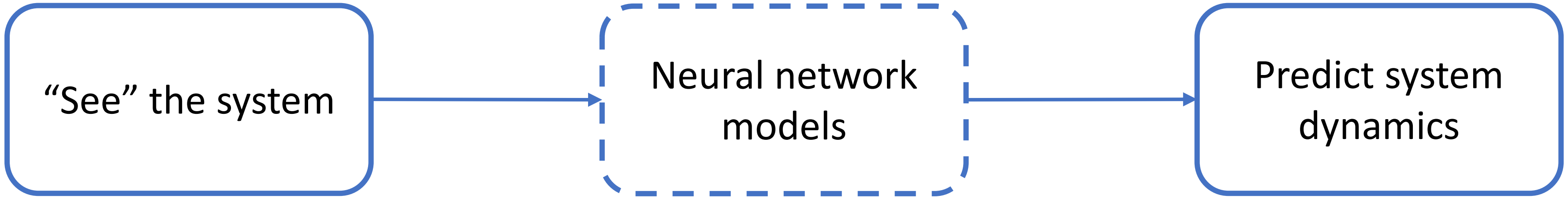
[1] Brunton, S.L., Proctor, J.L. and Kutz, J.N., 2016. Discovering governing equations from data by sparse identification of nonlinear dynamical systems.

[2] Greydanus S, Dzamba M, Yosinski J. Hamiltonian neural networks.

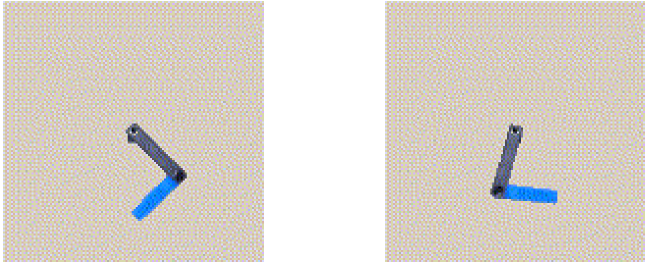
Modelling dynamical systems



“See” the system

Neural network models

Predict system dynamics



The predicted video looks good at the beginning, but it then becomes blur, heavily distorted, or plain background.

Prediction error accumulates in time: common challenge in video prediction[1].

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[1] Oprea S, Martinez-Gonzalez P, Garcia-Garcia A, Castro-Vargas JA, Orts-Escolano S, Garcia-Rodriguez J, Argyros A. A review on deep learning techniques for video prediction. IEEE Transactions on Pattern Analysis and Machine Intelligence. 2020 Dec 15.

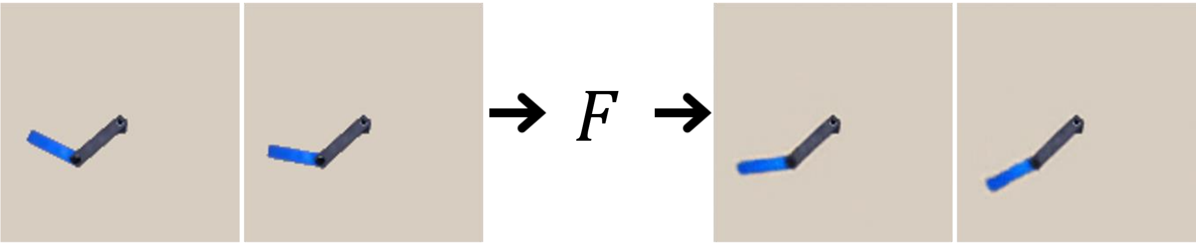
Goals

- Data: high-dimensional visual observations.
- Identify the number of state variables and extract them.
- Improve prediction quantity.

Low-dimensional state variables: how to discover?

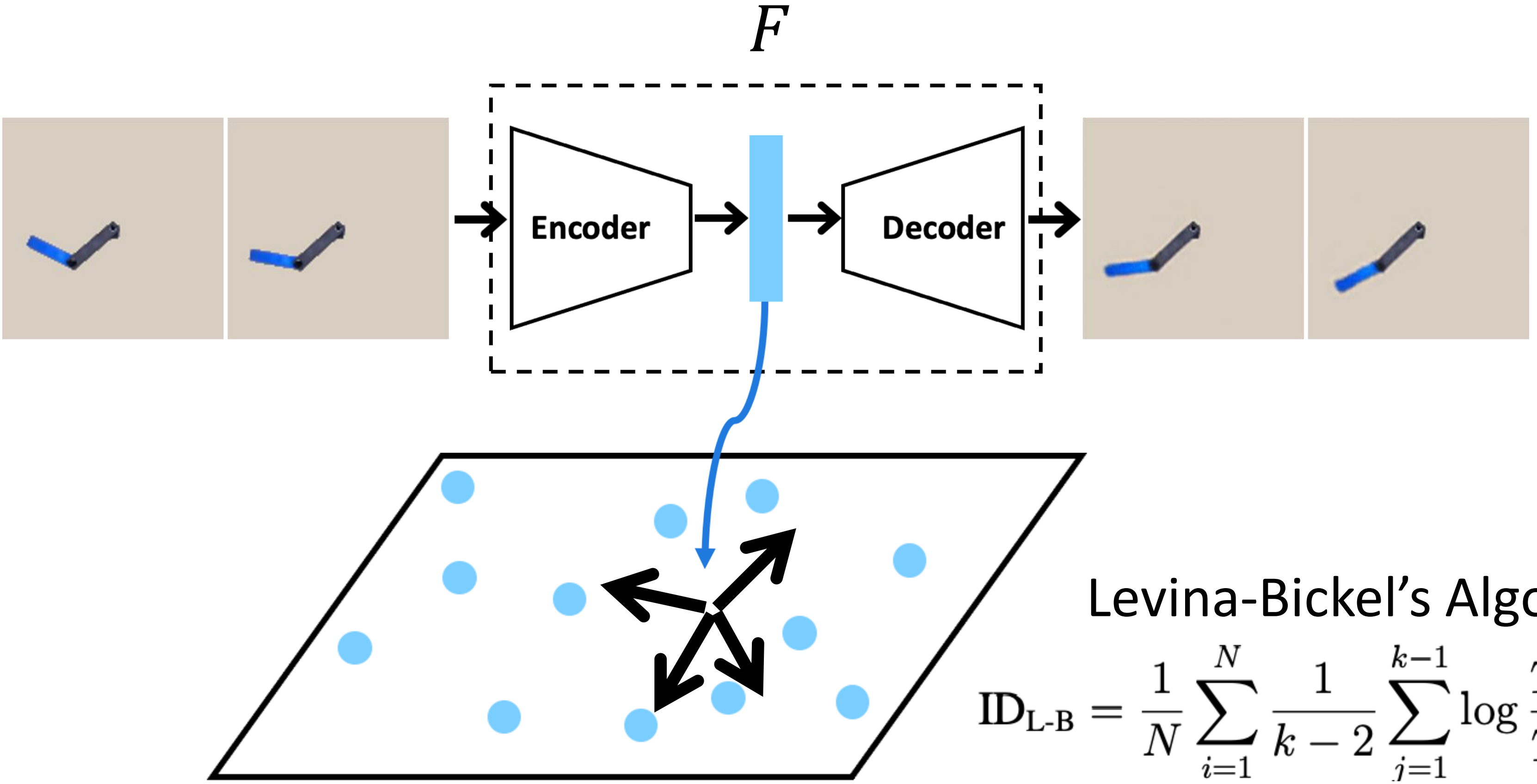
Mathematical Formulation

$$X_{t+dt} = F(X_t), \qquad t = 0, dt, 2dt, \dots$$
$$X_t \in S \subset \mathbb{R}^D$$



- dt: discrete time increment;
- X_t: system state;
- F: state evolution (dynamics);
- $\mathbb{R}^D$: high-dimensional ambient space, $D \sim 10^5$;
- S: low-dimensional manifold in the ambient space.
- Intrinsic dimension of S = number of state variables.

Identifying the number of state variables



Levina-Bickel's Algorithm

$$\mathrm{ID}_{\text{L-B}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{k-2}\sum_{j=1}^{k-1}\log\frac{T_k(\boldsymbol{L}^{(i)})}{T_j(\boldsymbol{L}^{(i)})}$$

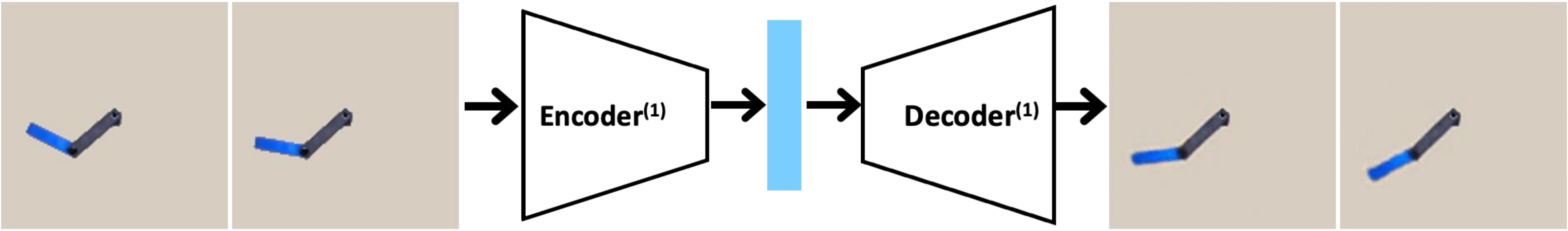
Geometric manifold learning problem

Identifying the number of state variables

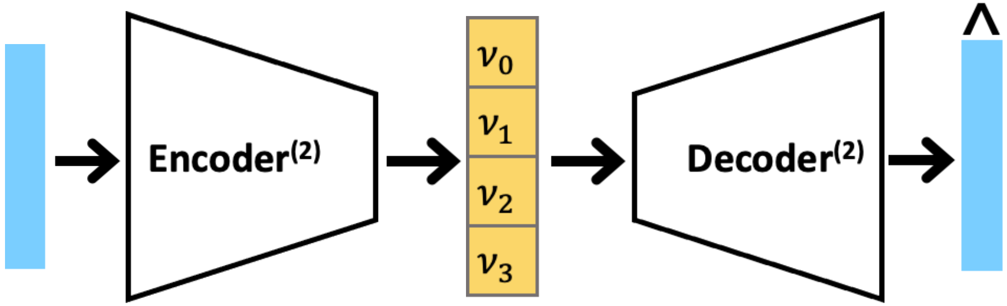
System	**ID from Raw Image**	**ID from Latent Vectors**	**Ground Truth**
Circular motion	3.67 (± 0.20)	2.19 (± 0.05)	2
Reaction diffusion	2.20 (± 0.13)	2.16 (± 0.14)	2
Single pendulum	3.65 (± 0.08)	2.05 (± 0.02)	2
Rigid double pendulum	7.08 (± 0.15)	4.71 (±0.03)	4
Swing stick	11.47 (± 0.38)	4.89 (± 0.33)	4
Elastic double pendulum	7.55 (± 0.15)	5.34 (± 0.20)	6
Air dancer	8.09 (± 0.19)	7.57 (± 0.13)	Unknown
Lava lamp	7.99 (± 0.41)	7.89 (±0.96)	Unknown
Fire	20.10 (± 0.50)	24.70 (± 2.02)	Unknown

Extracting state variables

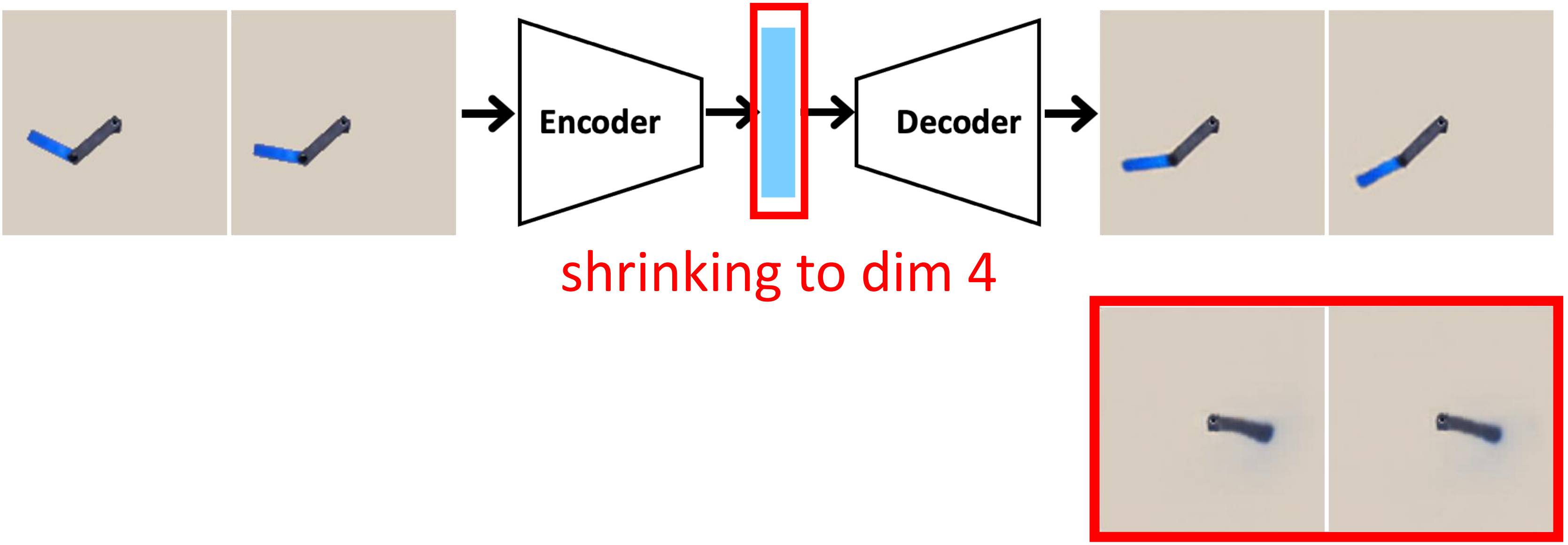
1st Stage



2nd Stage



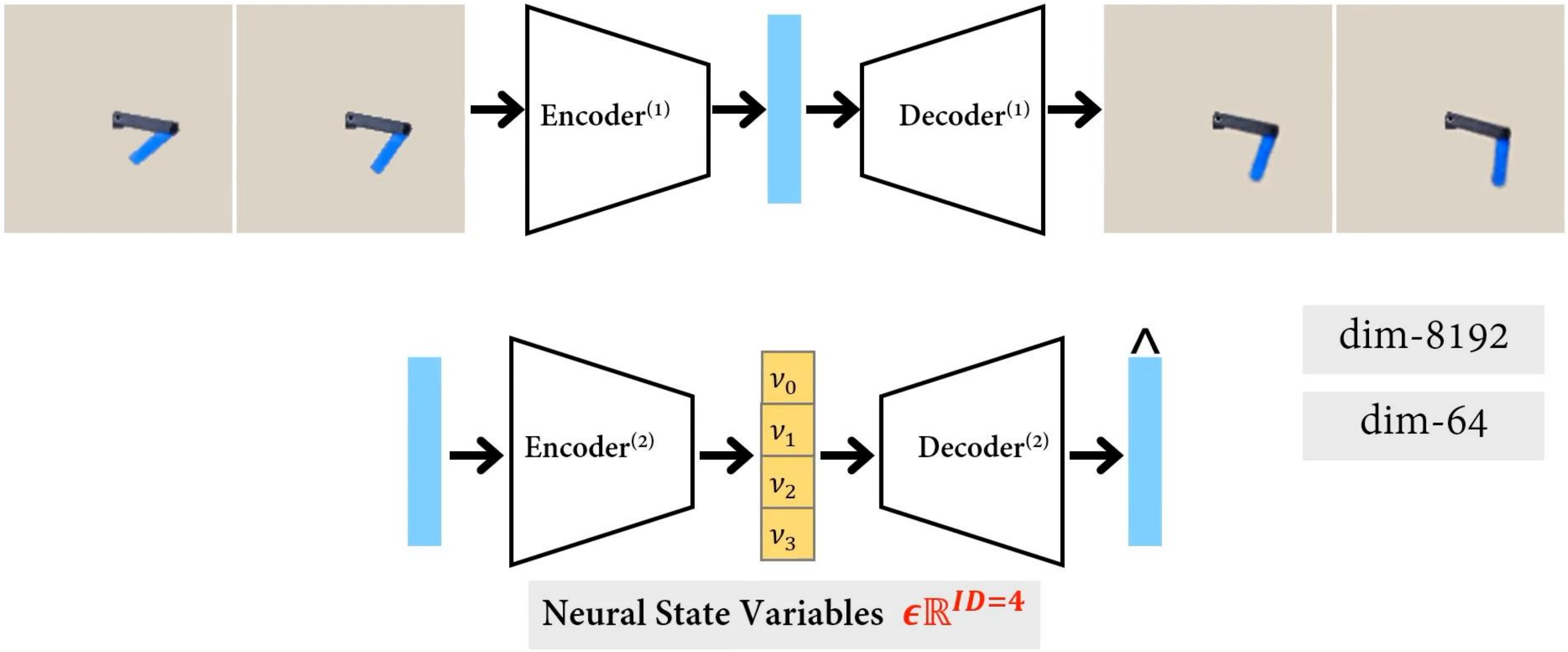
Extracting state variables



shrinking to dim 4

Long-term dynamics prediction: how to improve?

Prediction schemes through different intermediate dimensions



Encoder$^{(1)}$

Decoder$^{(1)}$

Encoder$^{(2)}$

v_0

v_1

v_2

v_3

Decoder$^{(2)}$

dim-8192

dim-64

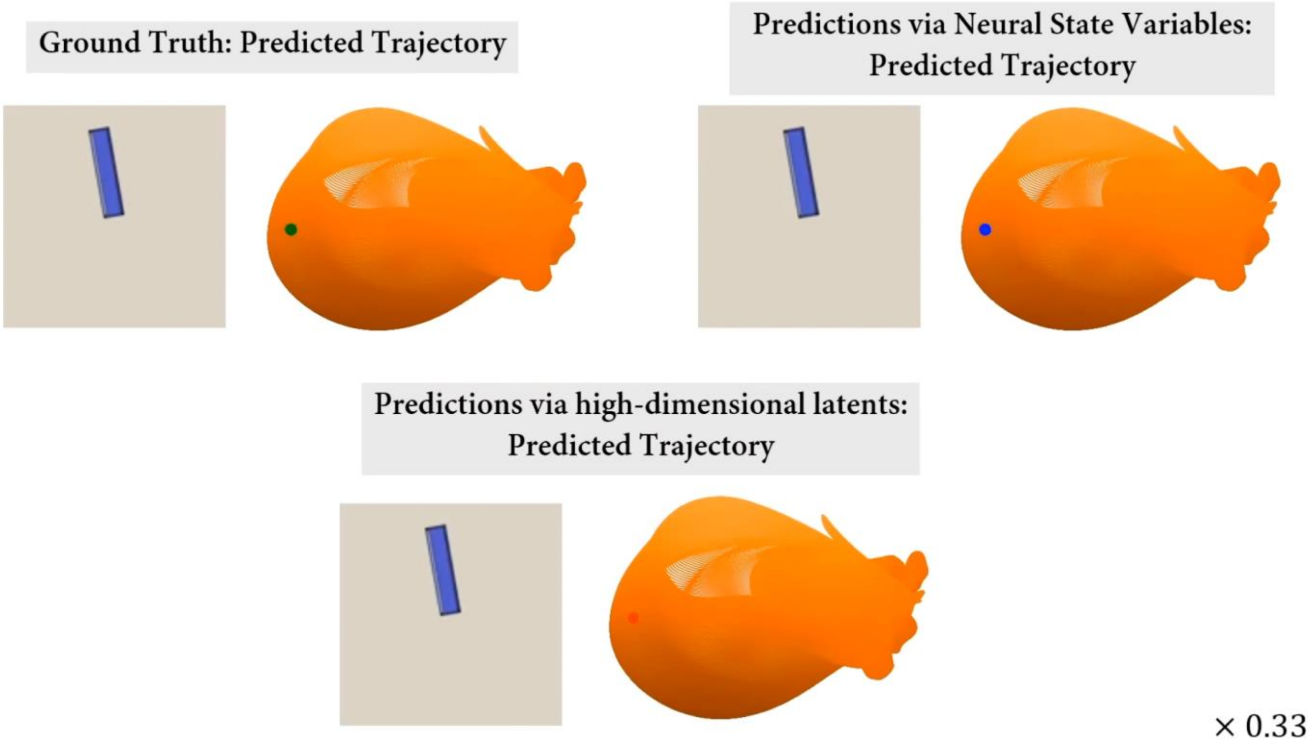
Neural State Variables $\in \mathbb{R}^{ID=4}$

Prediction stability

A scheme is **stable** if the prediction sequences always stay on the state manifold S.

A scheme is **unstable** if the prediction sequences can deviate from the state manifold.

Long-term prediction accuracy = one-step prediction accuracy + stability



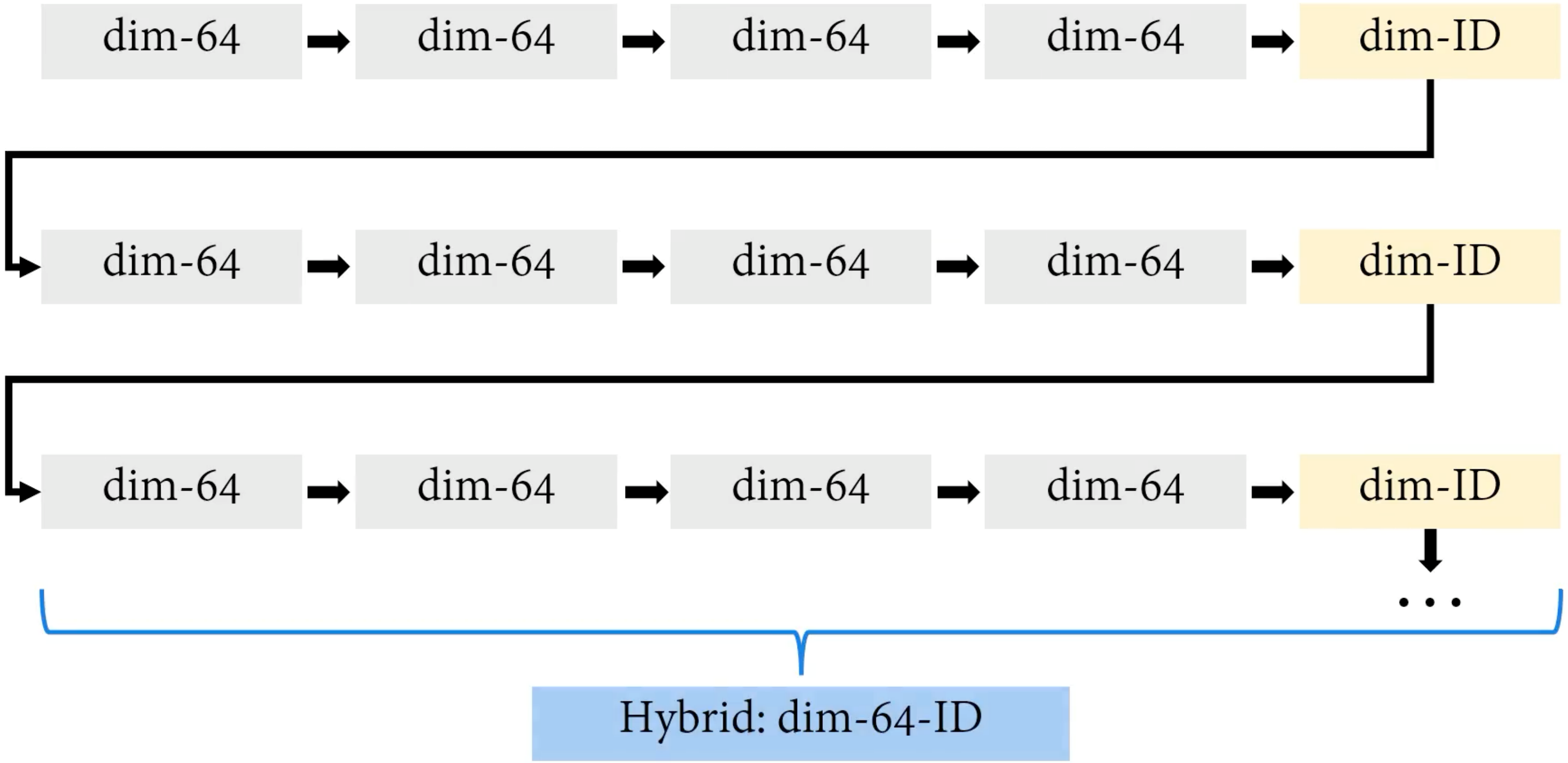
Ground Truth: Predicted Trajectory

Predictions via Neural State Variables: Predicted Trajectory

Predictions via high-dimensional latents: Predicted Trajectory

× 0.33

Quantitative evaluation: double pendulum



dim-64 → dim-64 → dim-64 → dim-64 → dim-ID

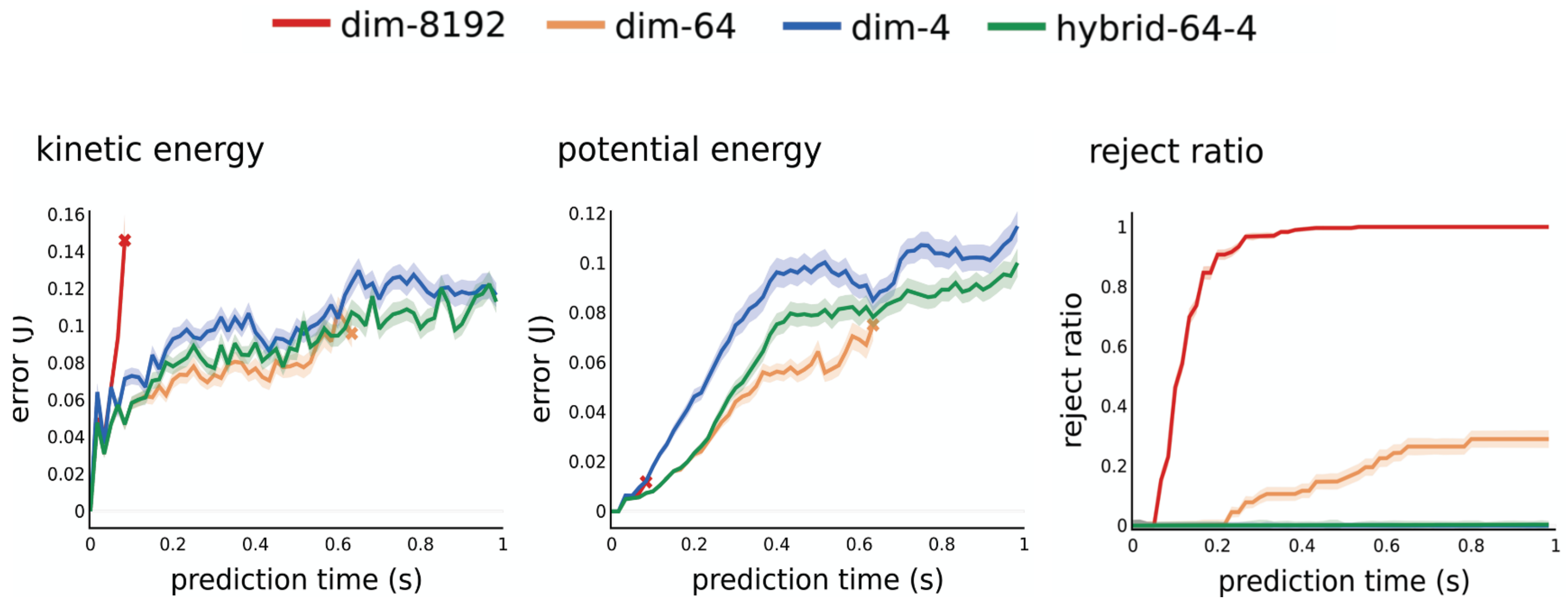
dim-64 → dim-64 → dim-64 → dim-64 → dim-ID

dim-64 → dim-64 → dim-64 → dim-64 → dim-ID

…

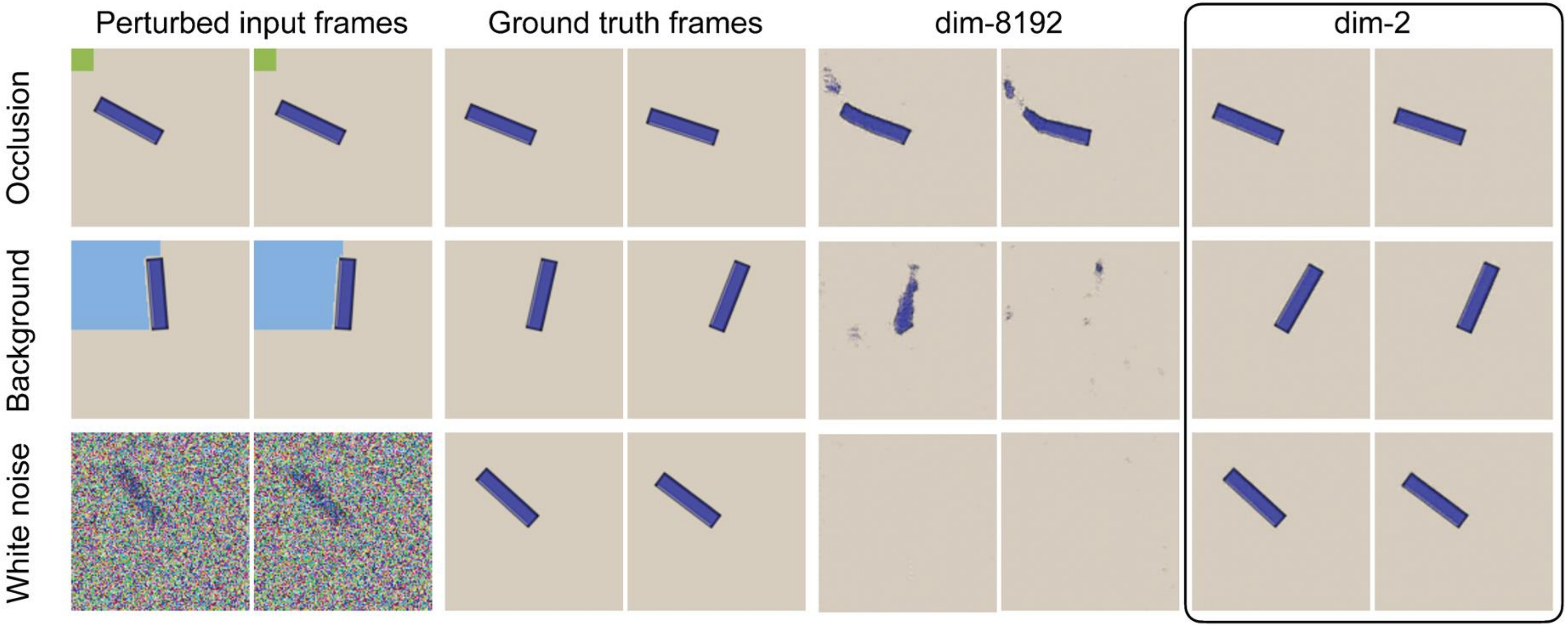
Hybrid: dim-64-ID

Quantitative evaluation: double pendulum



- The dim-4 scheme and the hybrid scheme are more stable.
- The hybrid scheme produces more accurate predictions.

Robustness


Perturbed input frames
Ground truth frames
dim-8192
dim-2
Occlusion
Background
White noise

Summary

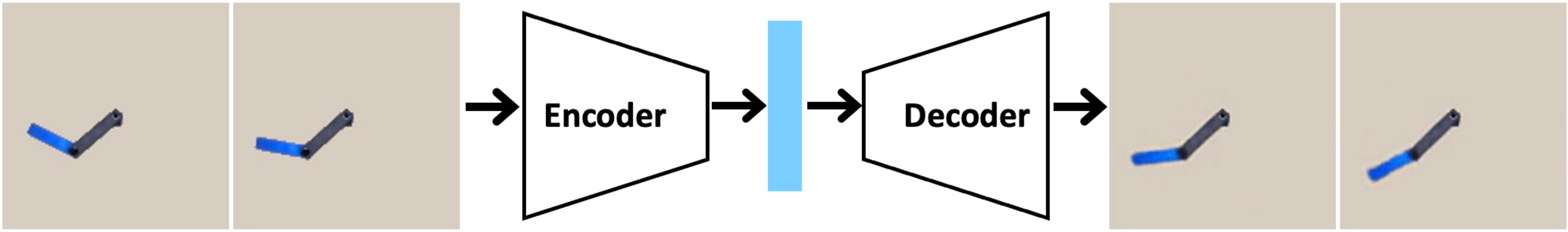
- A systematic framework to discover low-dimensional state variables from high-dimensional observations.

- The framework combines deep learning techniques and manifold learning algorithms.

- The learned low-dimensional state variables can help improve the long-term prediction stability.

- Key insights: **dimensionality and stability**.

Underlying Mathematics



- The network: $\hat{F}(x;\theta) = \varphi_D \circ \varphi_E(x;\theta)$, θ represents network parameters.
- The ojective function $L(\theta) = \frac{1}{2}\sum_{x\in\mathcal{S}} \left|\left|\hat{F}(x;\theta) - F(x)\right|\right|^2$.
- It is a high-dimensional non-convex optimization problem, there could be infinitely many global minimizers.
- The training dynamics (e.g., SGD) may provide implicit regularization. The effect of regularization depends on intermediate latent dimension.

Thanks!

Paper website:
https://www.cs.columbia.edu/~bchen/neural-state-variables/

Code base:
https://github.com/BoyuanChen/neural-state-variables