IFF: A Super-resolution Algorithm for Mulitple Measurements

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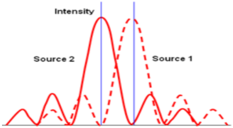
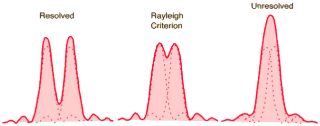
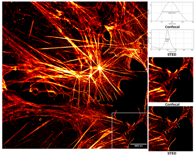
Rayleigh Limit

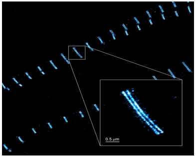
1. Resolution Limit: the minimum distance between two sources such that they can be distinguished.
2. Rayleigh criterion (1879): two point sources are regarded as just resolved when the principal diffraction maximum of one image coincides with the first minimum of the other.
3. In 1-D system, take the point spread function to be $\left(\frac{\sin \Omega x}{\Omega x}\right)^2$, then the Rayleigh Limit is $\frac{\pi}{\Omega}$ where Ω is the cutoff frequency.





Super-resolution Microscopy





- Stefan W. Hell and Jan Wichmann (1994): Stimulated emission depletion microscopy (STED).

  Selectively deactivating fluorophores to minimize the illuminated area

- Michael J. Rust, M. Bates and X. Zhuang (2006): Stochastic optical reconstruction microscopy (STORM).

  Stochastically activating the individual photoactivatable fluorophores.

Super-resolution Algorithms

1. Single snapshot:
   - Subspace method

     MUSIC Method, Matrix Pencil method, etc.
   - Convex optimization based method

     Total variation minimization, atomic norm minimization, etc.
2. Multiple measurements:
   - Subspace based method

     Aligned MUSIC/MP Method, etc.
   - Convex optimization based method

     Joint sparsity, ALOHA, etc

Theoretical Results

1. Donoho (1992): for point sources supported on a lattice with equal spacing (grid setting), the Minimax error of intensity recovery scales like $SRF^{\alpha}\sigma$ $(2n-1 \leq \alpha \leq 4n+1)$, where $SRF := \frac{\text{Rayleigh Limit}}{\text{grid spacing}}$;
2. L. Demanet and N. Nguyen (2015): The minimax error scales like $SRF^{2n-1}\sigma$ in the grid setting;
3. W. Li and W. Liao (2018) and D. Batenkov, L. Demanet (2019): The minimax error in multi-cluster case scales like $SRF^{2k-1}\sigma$ in the grid setting.
4. D. Batenkov, G. Goldman and Y. Yomdin (2019): The minimax error of intensity recovery scales as $SRF^{2n-1}\sigma$, while for support recovery scales as $SRF^{2n-2}\frac{\sigma}{\Omega}$ (off-the-grid).

Theoretical Results

1. P.Liu and H.Zhang (2021):

$$\mathcal{D}_{num} \sim \frac{C}{\Omega}\left(\frac{1}{SNR}\right)^{\frac{1}{2n-2}}, \tag{1}$$

$$\mathcal{D}_{supp} \sim \frac{C}{\Omega}\left(\frac{1}{SNR}\right)^{\frac{1}{2n-1}}. \tag{2}$$

2. P.Liu, S.Yu, etc (2022):

$$\mathcal{D}_{recon} \sim \frac{C}{\Omega}\left(\frac{1}{\sigma_{\infty,\min}(L)}\frac{1}{SNR}\right)^{\frac{1}{n}}$$

Mathematical Model

Collection of point sources:

$$\mu = \sum_{j=1}^{n} a_j \delta_{y_j}, \;\; y_j \in [-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega}].$$

Noisy measurements in frequency domain:

$$Y_t(\omega) = \mathcal{F}(\mu \cdot I_t) + W_t, \quad ||W_t||_\infty < \sigma, \; t = 1, \cdots, T,$$

$$Y_t(\omega_k) = \sum_{j=1}^{n} a_j I_t(y_j) e^{iy_j\omega_k} + W_t(\omega_k), \quad \omega_{-K}, \cdots, \omega_K \in [-\Omega, \Omega].$$

Assumption: $K \geq n$ and $T \geq n$.

IFF Method

Iteratively Focusing-localization and Filtering:

- Source focusing and localization
- Annihilating filter based source removal

Feature: Reconstruct point sources one by one in an iterative manner.

Source Focusing and Localization

Mathematical Model in Matrix Form

$$\begin{pmatrix} Y_1(\omega_{-K}) & \cdots & Y_1(\omega_K) \\ \vdots & & \vdots \\ Y_T(\omega_{-K}) & \cdots & Y_T(\omega_K) \end{pmatrix} = \begin{pmatrix} I_1(y_1) & \cdots & I_1(y_n) \\ \vdots & & \vdots \\ I_T(y_1) & \cdots & I_T(y_n) \end{pmatrix} \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \begin{pmatrix} e^{iy_1\omega_{-K}} & \cdots & e^{iy_1\omega_K} \\ \vdots & & \vdots \\ e^{iy_n\omega_{-K}} & \cdots & e^{iy_n\omega_K} \end{pmatrix} + \begin{pmatrix} W_1(\omega_{-K}) & \cdots & W_1(\omega_K) \\ \vdots & & \vdots \\ W_T(\omega_{-K}) & \cdots & W_T(\omega_K) \end{pmatrix}.$$

We denote it as

$$Y = LAE + W$$

To focus on the j-th source, we write

$$Y = LU_j \cdot U_j^{-1} AE + W,$$

where U_j is the permutation matrix.

Source Focusing and Localization

Observation: Suppose we apply QR decomposition to LU_j, we have

$$Q^* Y = \begin{pmatrix} R \\ 0 \end{pmatrix} U_j^{-1} AE + Q^* W. \tag{3}$$

The n-th row of (3), denoted as $\tilde{Y}_j$, gives

$$\begin{aligned} \tilde{Y}_j &= \left(\sum_{t=1}^{T} q_{tn} Y_t \left(\omega_{-K} \right), \cdots, \sum_{t=1}^{T} q_{tn} Y_t \left(\omega_{K} \right) \right) \\ &\triangleq R_{nn} \cdot a_j \left(e^{iy_j \omega_{-K}}, \cdots, e^{iy_j \omega_{-K}} \right) + \tilde{W}_j. \end{aligned} \tag{4}$$

Source Focusing and Localization

- Source Focusing: Solve an optimization problem for linear combination coefficient $\{q_{tn}\}_{t=1}^{T}$.
- Localization: Apply subspace method to reconstruct the source position y_j.

Output: $\{\hat{y}_p\}_{p=1}^{P}$, $P \leq n$.

Annihilating Filter Based Source Removal

Example:

Suppose, we have the measurement

$$Y = \left(ae^{iz\omega_{-K}}, ae^{iz\omega_{-K+1}}, \cdots, ae^{iz\omega_{K}}\right),$$

We define $F = \left(1, -e^{iz\frac{\Omega}{K}}\right)$, the discrete convolution gives

$$Y * F = \left(ae^{-iz\Omega}, 0, 0, \cdots, 0, -ae^{iz\frac{K+1}{K}\Omega}\right).$$

Annihilating Filter Based Source Removal

For $\{\hat{y}_p\}_{p=1}^{P}$, we define the annihilating filter as

$$F = \left(1, -e^{i\hat{y}_1 \frac{\Omega}{K}}\right) * \left(1, -e^{i\hat{y}_2 \frac{\Omega}{K}}\right) * \cdots * \left(1, -e^{i\hat{y}_P \frac{\Omega}{K}}\right).$$

The measurements after filtering are

$$Y_t' = (Y_t * F)\,[P+1 : 2K+1], \quad t = 1, \cdots, T.$$

- Source Removal: Filter all the recovered source from the original measurement for further processing.

Theoretical Grounds

Recall the measurement after perfect source focusing:

$$\tilde{Y}_j = R_{nn} \cdot a_j \left(e^{iy_j \omega_{-K}}, \cdots, e^{iy_j \omega_{-K}} \right) + \tilde{W}_j, \ \|\tilde{W}\|_\infty \le \sigma' \le \sqrt{T}\sigma.$$

Proposition

For L_{ij} are i.i.d. subgaussian random variables with $\mathbb{E}L_{ij} = 0$ and $\|L_{ij}\|_{\psi_2} \le B$, for any $t > 0$, we have

$$\begin{aligned} &\mathbb{P}\left(\left|R_{nn}^2 - (T - n + 1)\, v^2\right| > t\right) \\ &\le 2\exp\left(-c\min\left(\frac{t^2}{B^4\,(T - n + 1)}, \frac{t}{B^2}\right)\right), \end{aligned} \tag{5}$$

If we have enough measurements, $R_{nn} \sim O(\sqrt{T})$ with high probability.

Theoretical Grounds

In the perfect focusing case, $M = |R_{nn} \cdot a_j|$.

Theorem

Let $n \geq 2$, a collection of point sources $\{\delta_{y_j}\}_{j=1}^n$ is supported on $[-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega}]$ satisfying the following condition:

$$\tau = \min_{p \neq q} |y_p - y_q| \geq \frac{3.03\pi e}{\Omega} \left(\frac{\sigma'}{M}\right)^{\frac{1}{n}}. \tag{6}$$

If $\{\delta_{y_j}\}_{j=1}^n$ is σ'-admissible to $\mu = M\delta_y$, then

$$\min_{1 \leq j \leq n} |y - y_j| < \frac{\tau}{2}.$$

Theoretical Grounds

Proposition

For given $0 < \sigma' < M$*, and integer* $n \geq 2$*, let*

$$\tau = \frac{0.96e^{-\frac{3}{2}}}{\Omega}\left(\frac{\sigma'}{M}\right)^{\frac{1}{n}}. \tag{7}$$

For uniformly separated point sources $\{\delta_{y_j}\}_{j=1}^{n}$ *with distance* τ*. There exist* $y_k \in \{y_j\}_{j=1}^{n}$ *such that* $\mu = M\delta_k$*,* $\hat{\mu} = \sum_{j\neq k} \hat{a}_j\delta_{y_j}$ *satisfying* $\|[\mu] - [\hat{\mu}]\|_{\infty} < \sigma'$*.*

The above two results indicates that

$$\mathcal{D}_{comp} \sim \frac{C}{\Omega}\left(\frac{\sigma'}{M}\right)^{\frac{1}{n}}.$$

Numerical Experiments

Phase transition phenomenon of IFF Method

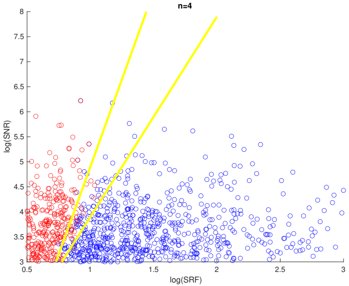


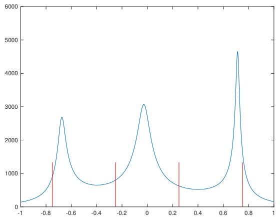
Figure: Plot of successful and unsuccessful point source reconstruction by IFF method in the parameter space $\log(SNR) - \log(SRF)$. Red one represents successful case and blue one represents unsuccessful case.

Numerical Experiments

Numerical behavior of IFF Method

Let $\Omega = 1$, $n = 4$, $\sigma = 1e-4$, $\mu = \delta_{-0.75} + \delta_{-0.25} + \delta_{0.25} + \delta_{0.75}$.

- For single snapshot:



- By IFF Method: We use 10 measurements each time and the mean of position is $(-0.7497, -0.2492, 0.2493, 0.7496)$ for 1000 times random experiments.

Conclusion

IFF Method

1. solves super-resolution problem with multiple measurements using one-by-one strategy,
2. circumvents the computation of singular-value decomposition for large matrices,
3. achieves stable reconstruction for point sources with a minimum separation distance that is close to the theoretical limit.

References

1. Fei, Zetao, and Hai Zhang. "IFF: A Super-resolution Algorithm for Multiple Measurements." arXiv preprint arXiv:2303.06617 (2023).
2. https://en.wikipedia.org/wiki/STED_microscopy
3. https://oni.wpengine.com/storm-microscopy

Thanks!