Spectral Structure of the Neumann-Poincaré Operator on thin domains

Joint work with Hyeonbae Kang (Inha University, Korea)
and Yoshihisa Miyanishi (Shinshu University, Japan)

Kazunori Ando (Ehime University, Japan)
The 6th Young Scholar Symposium, East Asia IPIA
2023, March 26

Single Layer Potential and NP Operator

Assume Ω be a simply connected, bounded domain with boundary $\partial\Omega$ in $\mathbb{R}^d$. Let $\Gamma(x)$ be the fundamental solution to the Laplacian $\triangle$.

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} \log|x|, & \text{if } d = 2, \\ -\frac{1}{4\pi} \frac{1}{|x|}, & \text{if } d = 3. \end{cases}$$

The single layer potential of $\varphi(x)$ on $\partial\Omega$ is defined by

$$\mathcal{S}_{\partial\Omega}[\varphi](x) = \int_{\partial\Omega} \Gamma(x-y)\varphi(y)\,\mathrm{d}\sigma(y), \quad x \in \mathbb{R}^d.$$

The Neumann-Poincaré operator $\mathcal{K}_{\partial\Omega}$ is a boundary integral operator on $\partial\Omega$.

$$\begin{aligned} \mathcal{K}_{\partial\Omega}[\varphi](x) &= \int_{\partial\Omega} \partial_{\nu_y}\Gamma(x-y)\,\varphi(y)\,\mathrm{d}\sigma(y) \\ &= \frac{1}{w_d}\int_{\partial\Omega} \frac{\langle y-x, \nu_y\rangle}{|x-y|^d}\varphi(y)\,\mathrm{d}\sigma(y), \quad x \in \partial\Omega, \end{aligned}$$

where $w_2 = 2\pi$ and $w_3 = 4\pi$, and ∂_{ν_y} is the outward normal derivative to $\partial\Omega$ w.r.t. y variable.

Symmetrization of NP Operator

Let $\mathcal{K}_{\partial\Omega}^*$ be the $L^2(\partial\Omega)$-adjoint operator of $\mathcal{K}_{\partial\Omega}$. It is well-known that the Plemelj's symmetrization principle holds:

$$\mathcal{S}_{\partial\Omega}\mathcal{K}_{\partial\Omega}^* = \mathcal{K}_{\partial\Omega}\mathcal{S}_{\partial\Omega}.$$

Denote by $H^{1/2} = H^{1/2}(\partial\Omega)$ the $1/2$-Sobolev space on $\partial\Omega$, and by $H^{-1/2} = H^{-1/2}(\partial\Omega)$ its dual space. It is known that there exists a domain on which $\mathcal{S}_{\partial\Omega} : H^{-1/2} \to H^{1/2}$ is not invertible.

By changing the definition of $\mathcal{S}_{\partial\Omega}$ on a finite dimensional space, we have $\mathcal{S}_{\partial\Omega}^{-1} : H^{1/2} \to H^{-1/2}$. We can define the following inner product on $H^{1/2}$:

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = -\left\langle \varphi, \mathcal{S}_{\partial\Omega}^{-1}[\psi] \right\rangle, \quad \varphi, \psi \in H^{1/2},$$

which induces $\|\cdot\|_{\mathcal{H}} \simeq \|\cdot\|_{1/2}$.

$H^{1/2}$ equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is a Hilbert space, which is denoted by $\mathcal{H}$. By the Plemelj's symmetrization principle, $\mathcal{K}_{\partial\Omega}$ is self-adjoint on $\mathcal{H}$:

$$\langle \varphi, \mathcal{K}_{\partial\Omega}[\psi] \rangle_{\mathcal{H}} = \langle \mathcal{K}_{\partial\Omega}[\varphi], \psi \rangle_{\mathcal{H}}, \quad \varphi, \psi \in H^{1/2}.$$

Spectral Properties of the NP Operator

These spectral properties of the NP operator on $\mathcal{H}$ are well-known:

1. The spectrum of the NP operator $\sigma(\mathcal{K}_{\partial\Omega}) \subset (-1/2, 1/2]$.
2. If $\partial\Omega$ is smooth, at least $C^{1,\alpha}$, then the NP operator $\mathcal{K}_{\partial\Omega}$ is compact.
3. In 2D case, if λ is an NP eigenvalue, then so is $-\lambda$, except the simple eigenvalue $\lambda = 1/2$.
4. The NP eigenvalues coincide with those on $L^2(\partial\Omega)$.
5. The NP spectrum $\sigma(\mathcal{K}_{\partial\Omega})$ is scale invariant.

Examples

Disk (2d): $x^2 + y^2 = 1$,

$$\lambda_n = \frac{1}{2}, 0 \text{ } (\infty\text{-multiplicities})$$

Ellipse (2d): $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$,

$$\lambda_n = \frac{1}{2}, \pm\frac{1}{2}\left(\frac{a-b}{a+b}\right)^n, \ n = 1, 2, \ldots,$$

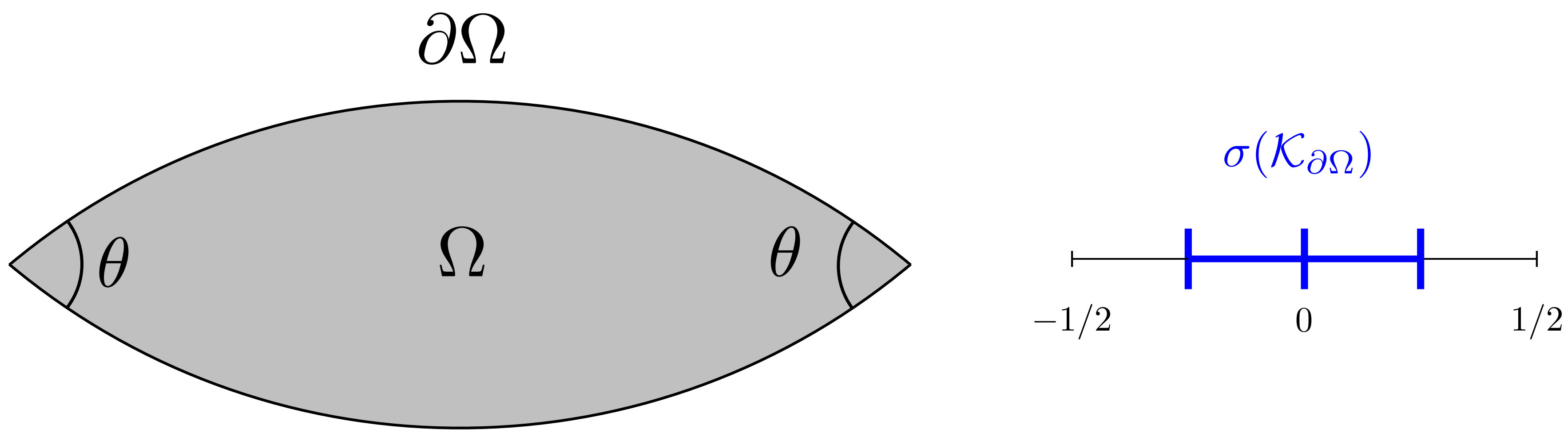
Sphere (3d): $x^2 + y^2 + z^2 = 1$,

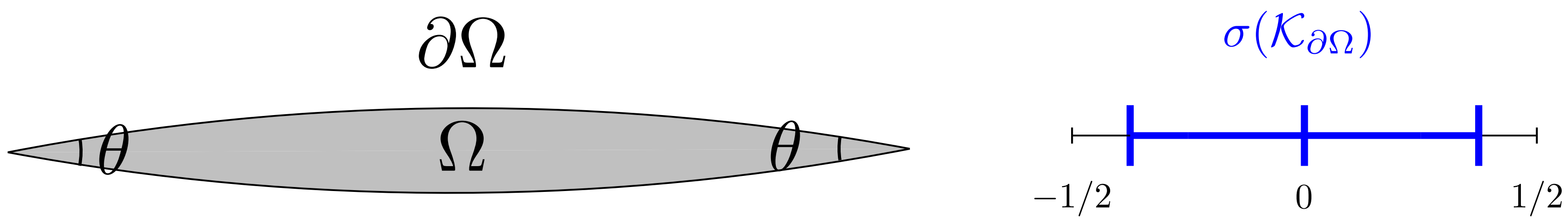
$$\lambda_n = \frac{1}{2\,(2n+1)}, \quad n = 0, 1, 2, \ldots,$$

with $2n+1$ multiplicity for each λ_n.

Known Results (thin convex lens shape domains)

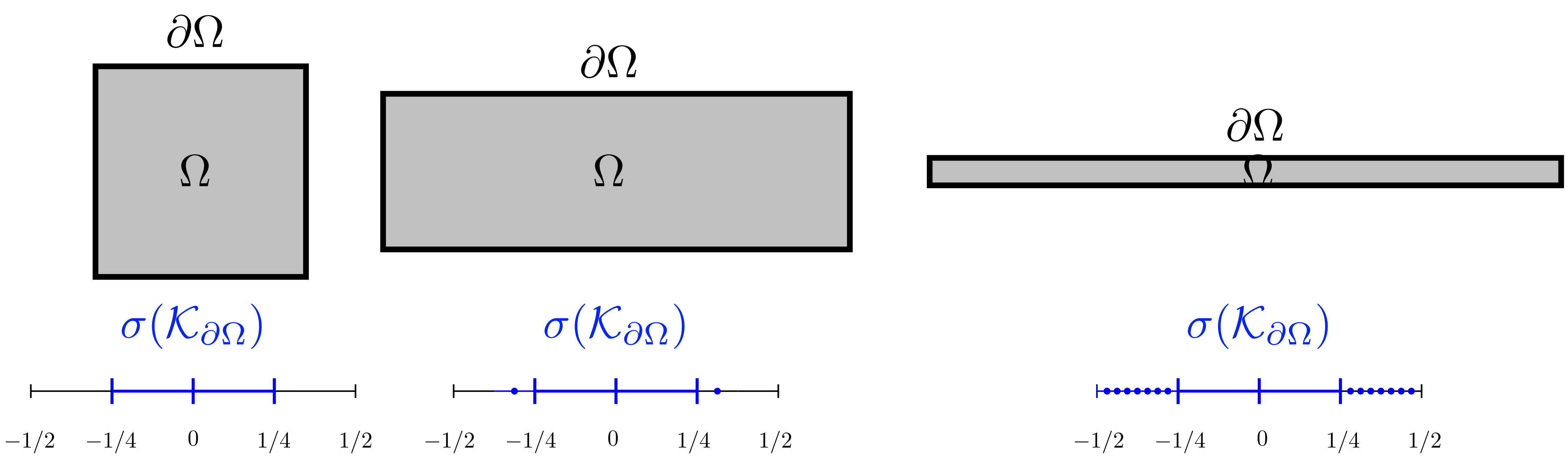
If the planar domain has corners, the essential spectrum of the NP operator is $\left[-\frac{1}{2}(1-\theta/\pi), \frac{1}{2}(1-\theta/\pi)\right]$, where θ is the smallest angle of the corners (Perfekt, Putinar (2014, 2017)).If Ω is an intersecting discs, $\sigma(\mathcal{K}_{\partial\Omega}) = \left[-\frac{1}{2}(1-\theta/\pi), \frac{1}{2}(1-\theta/\pi)\right]$ is absolutely continuous (Kang, Lim, Yu (2017)). The angle gets thiner ($\theta \to 0$), the spectrum approaches $[-1/2, 1/2]$.





Known Results (thin rectangle shape domains)

In the case of rectangular domains, the essential spectrum is $[-1/4, 1/4]$. If the rectangular domain gets thiner, there appear the eigenvalues outside the essential spectrum (Helsing, Kang, Lim (2017)).



We are interested in the spectrum of the NP operator as the domain gets thiner and thiner. We may expect that $\sigma(\mathcal{K}_{\partial\Omega})$ fills the interval $[-1/2, 1/2]$ in some sense.

Main Result 1 (2d thin domains)

Let Ω_R, $R \geq 1$, be a rectangle-shaped domain whose boundary is:

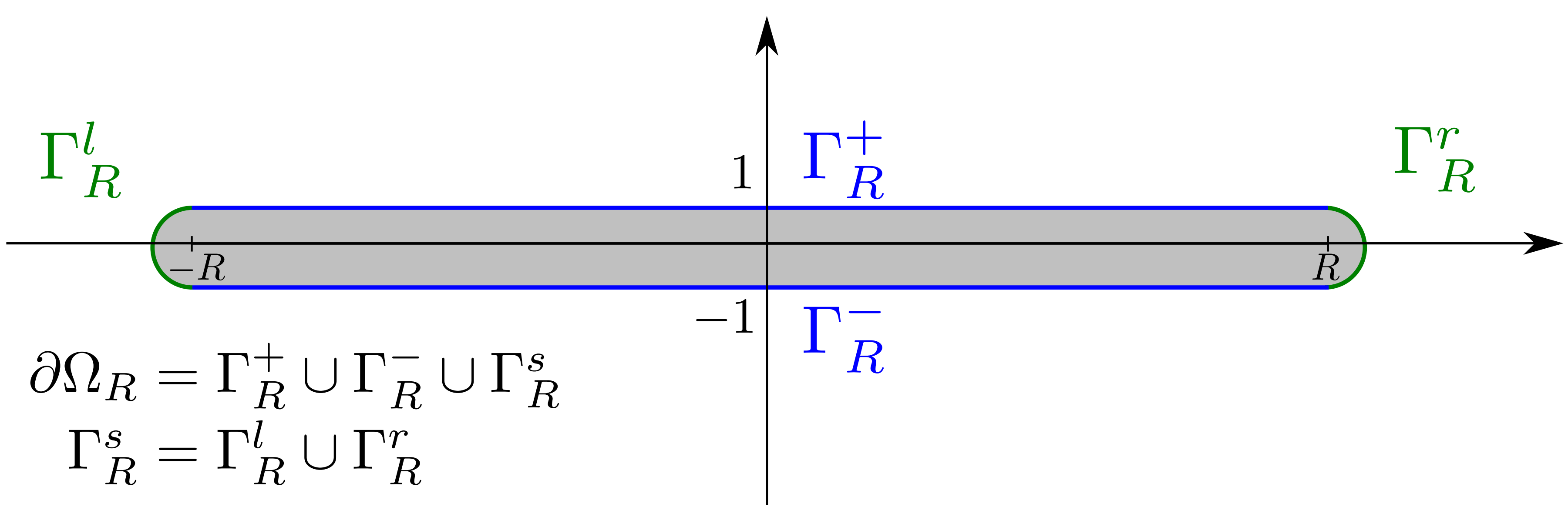
$$\partial\Omega_R = \Gamma_R^+ \cup \Gamma_R^- \cup \Gamma_R^s, \quad \Gamma_R^\pm = [-R, R] \times \{\pm 1\},$$

and the side Γ_R^s consists of the left and the right sides $\Gamma_R^s = \Gamma_R^l \cup \Gamma_R^r$, where Γ_R^l and Γ_R^r are curves connecting points $(\mp R, 1)$ and $(\mp, -1)$, respectively.

We assume that Γ_R^l and Γ_R^r are of any fixed shape independent of R and $\partial\Omega_R$ is Lipschitz continuous. We call R the aspect ratio of $\partial\Omega_R$.

Theorem 1. *If $\{R_j\}$ be an increasing sequence such that $R_j \to \infty$ as $j \to \infty$, then*

$$\overline{\cup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial\Omega_{R_j}})} = [-1/2, 1/2].$$



Test Functions

To prove the Theorem 1, we construct test functions to approximate the NP eigenfunction. For a given compactly supported f on $(-R, R)$, define

$$\varphi(x) = \varphi(x_1, x_2) = \begin{cases} f(x_1), & \text{if } x \in \Gamma_R^+ \cup \Gamma_R^- \\ 0, & \text{if } x \in \Gamma_R^s. \end{cases}$$

Put $\mathcal{K}_R = \mathcal{K}_{\partial\Omega_R}$ for simplicity. Note that $\mathcal{K}_R[\varphi](x_1, x_2)$ is written as

$$-\frac{1}{2\pi}\int_{\mathbb{R}} \frac{x_1 - 1}{(x_1 - y_1)^2 + (x_2 - 1)^2} f(y_1)dy_1 + \frac{1}{2\pi}\int_{\mathbb{R}} \frac{x_1 + 1}{(x_1 - y_1)^2 + (x_2 + 1)^2} f(y_1)dy_2.$$

Thus, if $(x_1, x_2) \in \Gamma_R^+ \cup \Gamma_R^-$ (i.e., $x_2 = \pm 1$),

$$\mathcal{K}_R[\varphi](x_1, x_2) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{2}{(x_1 - y_1)^2 + 2^2} f(y_1)dy_1 = \frac{1}{2}P_2 * f(x_1),$$

where P_t is the Poisson kernel on the half space,

$$P_t(x_1) = \frac{1}{\pi}\frac{t}{|x_1|^2 + t^2}, \quad t > 0.$$

For a given $\lambda \in (0, 1/2]$, we look for h such that

$$\lambda h - \frac{1}{2} P_2 * h = 0.$$

Using the Fourier transform $\widehat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x} f(x) dx$, we have the following, since $\hat{P}_t(\xi) = e^{-2\pi t|\xi|}$:

$$\left(\lambda - \frac{1}{2} e^{-4\pi|\xi|}\right) \widehat{h}(\xi) = 0.$$

Let $\xi_0 > 0$ such that $\lambda - \frac{1}{2} e^{-4\pi|\xi_0|} = 0$. Then, if we can construct $\widehat{f}_R \to \widehat{h} = \delta_{\xi_0}$, $\widehat{f}_R$ should be an approximate eigenfunction w.r.t. the eigenvalue λ.

We can construct such f_R as follows:

$$f_R(x) = e^{2\pi i \xi_0 x} (\chi\psi)(R^{-1}x),$$

where χ is a smooth cut-off function such that $\operatorname{supp} \chi \subset [-1/2, 1/2]$ and $\chi = 1$ on $[-1/4, 1/4]$, and $\widehat{\psi}$ is a non-negative compactly supported smooth function such that $\int_{\mathbb{R}} \widehat{\psi}(\xi) d\xi = 1$. We note that $R\widehat{\psi}(R(\xi - \xi_0))$ converges weakly to $\delta_{\xi_0}(\xi)$.

Outline of Proof of Main Theorem 1

We denote by $\|\cdot\|_{1/2}$ the Sobolev $1/2$-norm on $\mathbb{R}$. We can prove the followings.

Lemma 1.1 *For $\lambda \in (0, 1/2]$, let f_R be the function defined as above. Then, it holds that*

$$R^{1/2} \lesssim \|f_R\|_{1/2} \quad \text{and} \quad \left\| \lambda f_R - \frac{1}{2} P_2 * f_R \right\|_{1/2} \lesssim R^{-1/2}.$$

Proposition 1.1 *Let $\lambda \in (0, 1/2]$. There is a sequence $\varphi_R \in H^{1/2}(\partial\Omega_R)$ s.t.*

$$\lim_{R \to \infty} \frac{\|(\lambda I - \mathcal{K}_R)[\varphi_R]\|_{H^{1/2}(\partial\Omega_R)}}{\|\varphi_R\|_{H^{1/2}(\partial\Omega_R)}} = 0.$$

Theorem 1 is a immediate consequence of Proposition 1.1. In fact, if $\lambda \in (0, 1/2]$ and $\lambda \notin \overline{\cup_{j=1}^{\infty} \sigma(\mathcal{K}_{R_j})}$, then $\operatorname{dist}\left(\lambda, \overline{\cup_{j=1}^{\infty} \sigma(K_{R_j})}\right) > 0$. Thus, there is $C > 0$ independent of j such that

$$\|\varphi_{R_j}\|_{H^{1/2}(\partial\Omega_{R_j})} \leq C \left\| \left(\lambda I - \mathcal{K}_{R_j}\right)[\varphi_{R_j}] \right\|_{H^{1/2}(\partial\Omega_{R_j})},$$

which contradicts Proposition 1.1.

Lemma1.1 For $\lambda \in (0, 1/2]$, let f_R be the function defined as above. Then, it holds that

$$R^{1/2} \lesssim \|f_R\|_{1/2} \quad \text{and} \quad \left\| \lambda f_R - \frac{1}{2} P_2 * f_R \right\|_{1/2} \lesssim R^{-1/2}.$$

Outline of Proof. Note that $\widehat{f_R}(\xi) = R\widehat{(\chi\psi)}(R(\xi - \xi_0))$. Thus, we have

$$\|f_R\|_{1/2}^2 = R \int_{\mathbb{R}} \left(1 + \left|\frac{\xi}{R} + \xi_0\right|\right) \left|\widehat{(\chi\psi)}(\xi)\right|^2 d\xi \geq R \int_{\mathbb{R}} \left|\widehat{(\chi\psi)}(\xi)\right|^2 d\xi.$$

On the other hand, $\mathcal{F}(\lambda f_R - \frac{1}{2} P_2 * f_R)(\xi) = (\lambda - \frac{1}{2} e^{-4\pi|\xi|} R\widehat{(\chi\psi)})(R(\xi - \xi_0))$. We split $\left\|\lambda f_R - \frac{1}{2} P_2 * f_R\right\|_{1/2}^2 = I + II$ as follows: if $|\xi| \leq \sqrt{R}$, $\left|\lambda - \frac{1}{2} e^{-4\pi\left|\frac{\xi}{R} + \xi_0\right|}\right| \lesssim |\xi|/R$, so

$$I := R \int_{|\xi| \leq \sqrt{R}} \left(1 + \left|\frac{\xi}{R} + \xi_0\right|\right) \left|\lambda - \frac{1}{2} e^{-4\pi\left|\frac{\xi}{R} + \xi_0\right|}\right|^2 \left|\widehat{(\chi\psi)}(\xi)\right|^2 d\xi \lesssim R^{-1},$$

and

$$II \lesssim R \int_{|\xi| > R} (1 + |\xi|)^{1-2N} d\xi \lesssim R^{1-N}.$$

Proposition 1.1 Let $\lambda \in (0, 1/2]$. Then, there exists a sequence $\varphi_R \in H^{1/2}(\partial\Omega_R)$ such that

$$\lim_{R\to\infty} \frac{\|(\lambda I - \mathcal{K}_R)[\varphi_R]\|_{H^{1/2}(\partial\Omega_R)}}{\|\varphi_R\|_{H^{1/2}(\partial\Omega_R)}} = 0.$$

Outline of Proof. Define φ_R on $\partial\Omega_R$ by

$$\varphi_R(x_1, x_2) = \begin{cases} f_R(x_1), & \text{if } x \in \Gamma_R^+ \cup \Gamma_R^-, \\ 0, & \text{if } x \in \Gamma_R^s. \end{cases}$$

By Lemma 1.1, we only have to show $\|(\lambda I - \mathcal{K}_R)[\varphi_R]\|_{H^{1/2}(\partial\Omega_R)} \lesssim 1$.

We choose a constant $C > 0$ independent of R such that

$$\Gamma_R^l \subset \{(x_1, x_2) : x_1 < -R + C\}, \quad \Gamma_R^r \subset \{(x_1, x_2) : x_1 > R - C\}.$$

Let $\zeta_1(x_1, x_2) = \zeta_1(x_1)$ be a smooth function such that $\operatorname{supp} \zeta_1 \subset (-R+C, R-C)$ and $\zeta_1 = 1$ on $[-R + 2C, R - 2C]$, and let $\zeta_2 = 1 - \zeta_1$. Then we have

$$\|(\lambda I - \mathcal{K}_R)[\varphi_R]\|_{H^{1/2}(\partial\Omega_R)} \le \sum_{j=1}^{2} \|\zeta_j (\lambda I - \mathcal{K}_R)[\varphi_R]\|_{H^{1/2}(\partial\Omega_R)}.$$

We have

$$\|\zeta_1(\lambda I - \mathcal{K}_R)[\varphi_R]\|_{H^{1/2}(\partial\Omega_R)} = \left\|\zeta_1\left(\lambda f_R - \frac{1}{2}(P_2 * f_R)\right)\right\|_{1/2} \lesssim R^{-1/2}.$$

Let $\Gamma := \partial\Omega_R \cap \{(x_1, x_2) : x_1 < -R + C$ or $x_1 > R - C\}$. To estimate the second term, we use the following characterization of $H^{1/2}(\Gamma)$:

$$\|h\|^2_{H^{1/2}(\Gamma)} = \|h\|^2_{L^2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|h(x) - h(z)|^2}{|x - z|^2} d\sigma(x) d\sigma(z).$$

Note that $\zeta_2(\lambda I - \mathcal{K}_R)[\varphi_R] = \zeta_2 \mathcal{K}_R[\varphi_R]$. If $x, z \in \operatorname{supp}(\zeta_2)$ and $y \in \operatorname{supp}(\varphi_R)$,

$$|x - z| \lesssim 1, \quad |x - y| \gtrsim R, \quad |z - y| \gtrsim R.$$

Thus, $|k_R(x, y)| \lesssim R^{-1}$. So, $\|\zeta_2 \mathcal{K}_R[\varphi_R]\|^2_{L^2(\Gamma)} \lesssim R^{-1} \int_{\mathbb{R}} |f_R(x_1)|^2 dx_1 \lesssim 1$. We also have$|\zeta_2(x) k_R(x, y) - \zeta_2(z) k_R(z, y)| \leq R^{-1} |x - z|$. Then it follows

$$\int_\Gamma \int_\Gamma \frac{|\zeta_2(x)\mathcal{K}_R[\varphi_R](x) - \zeta_2(z)\mathcal{K}_R[\varphi_R](z)|^2}{|x - z|^2} d\sigma(x) d\sigma(z) \lesssim 1.$$

Main Result 2 (3d thin domains)

Let U be a bounded domain with the Lipschitz continuous boundary ∂U, and let $U_R := RU$ be the dilation of R, $R \geq 1$. We assume that U contains the unit disk centered at 0. Let $\partial\Omega_R$ be the 3D thin domain

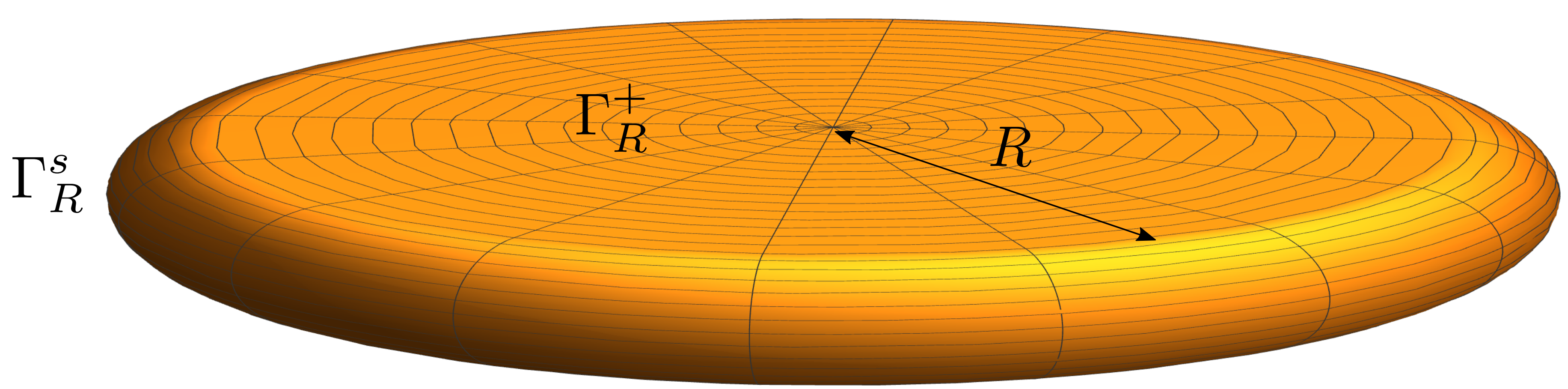
$$\partial\Omega_R = \Gamma_R^+ \cup \Gamma_R^- \cup \Gamma_R^s, \quad \Gamma_R^\pm = \{(x_1, x_2, x_3) : x' = (x_1, x_2) \in U_R, x_3 = \pm 1\},$$

and the side Γ_R^s connects Γ_R^+ and Γ_R^-.

The side Γ_R^s is of any fixed shape independent of R, and the boundary $\partial\Omega_R$ is smooth.

Theorem 2. *If $\{R_j\}$ be an increasing sequence such that $R_j \to \infty$ as $j \to \infty$, then*

$$\overline{\cup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial\Omega_{R_j}})} = [-1/2, 1/2].$$



Test Functions

We can prove Theorem 2 in the same way as Theorem 1. We construct test functions to approximate eigenvalues as follows:

$$\varphi_R(x) = \varphi_R(x', x_3) := \begin{cases} f_R(x'), & \text{if } x \in \Gamma_R^+ \cup \Gamma_R^-, \\ 0, & \text{if } x \in \Gamma_R^s. \end{cases}$$

Put $\mathcal{K}_R$ for simplicity. $\mathcal{K}_R[\varphi](\varphi)$ is written as

$$-\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{(x_3 - 1)\, f_R(y')dy'}{\left[\, |x' - y'|^2 + (x_3 - 1)^2 \,\right]^{3/2}} + \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{(x_3 + 1)\, f_R(y')dy'}{\left[\, |x' - y'|^2 + (x_3 + 1)^2 \,\right]^{3/2}}.$$

Thus, if $x \in \Gamma_R^+ \cup \Gamma_R^-$, then

$$\mathcal{K}_R[\varphi_R](x) = \frac{1}{2} \left(P_2 * f_R\right)(x'),$$

where P_t is the Poisson kernel on the half space in $\mathbb{R}^{2+1}$,

$$P_t(x') = \frac{1}{2\pi} \frac{t}{\left(|x'|^2 + t^2 \right)^{3/2}}, \quad t > 0.$$

For $\lambda \in (0, 1/2]$, we choose $\xi_0 \in \mathbb{R}^2$ so that

$$\lambda - \frac{1}{2} e^{-4\pi|\xi_0|} = 0.$$

We define

$$f_R(x) := e^{2\pi i \xi_0 x} (\chi\eta)(R^{-1}x),$$

where χ is a smooth cut-off function such that $\operatorname{supp} \chi \subset B_{1/2}$ and $\chi = 1$ on $B_{1/4}$, and $\widehat{\psi}$ is a non-negative compactly supported smooth function such that $\int_{\mathbb{R}^2} \widehat{\psi}(\xi) d\xi = 1$. We note that $R^2 \widehat{\psi}(R(\xi - \xi_0))$ converges weakly to $\delta_{\xi_0}(\xi)$. We obtain the followings.

Lemma 2.1 *For $\lambda \in (0, 1/2]$, let f_R be the function defined as above. Then it holds that*

$$R \lesssim \|f_R\|_{1/2}, \qquad \textit{and} \qquad \left\| \lambda f_R - \frac{1}{2} P_2 * f_R \right\|_{1/2} \lesssim 1.$$

Proposition 2.1 *Let $\lambda \in (0, 1/2]$. There is a sequence $\varphi_R \in H^{1/2}(\partial\Omega_R)$ such that*

$$\lim_{R \to \infty} \frac{\|(\lambda I - \mathcal{K}_R)[\varphi_R]\|_{H^{1/2}(\partial\Omega_R)}}{\|\varphi_R\|_{H^{1/2}(\partial\Omega_R)}} = 0.$$

Then we can prove that

$$\overline{\cup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial\Omega_{R_j}})} \supset (0, 1/2]$$

in the same as in Theorem 1. For $\lambda \in [-1/2, 0)$, define

$$\varphi_R(x) = \varphi_R(x', x_3) := \begin{cases} f_R(x'), & \text{if } x \in \Gamma_R^+, \\ -f_R(x'), & \text{if } x \in \Gamma_R^-, \\ 0, & \text{if } x \in \Gamma_R^s. \end{cases}$$

Then on can see as before that

$$\mathcal{K}_R[\varphi_R](x) = \begin{cases} -\frac{1}{2}(P_2 * f_R)(x'), & \text{if } x \in \Gamma_R^+, \\ \frac{1}{2}(P_2 * f_R)(x'), & \text{if } x \in \Gamma_R^-. \end{cases}$$

Thus,

$$\lambda\varphi_R(x) - \mathcal{K}_R[\varphi_R](x) = \begin{cases} \lambda f_R(x') + \frac{1}{2}(P_2 * f_R)(x'), & \text{if } x \in \Gamma_R^+, \\ -\lambda f_R(x') - \frac{1}{2}(P_2 * f_R)(x'), & \text{if } x \in \Gamma_R^-. \end{cases}$$

The rest of the proof is the same as that in the case for $\lambda \in (0, 1/2]$.

Main Result 3 (3d thin domains)

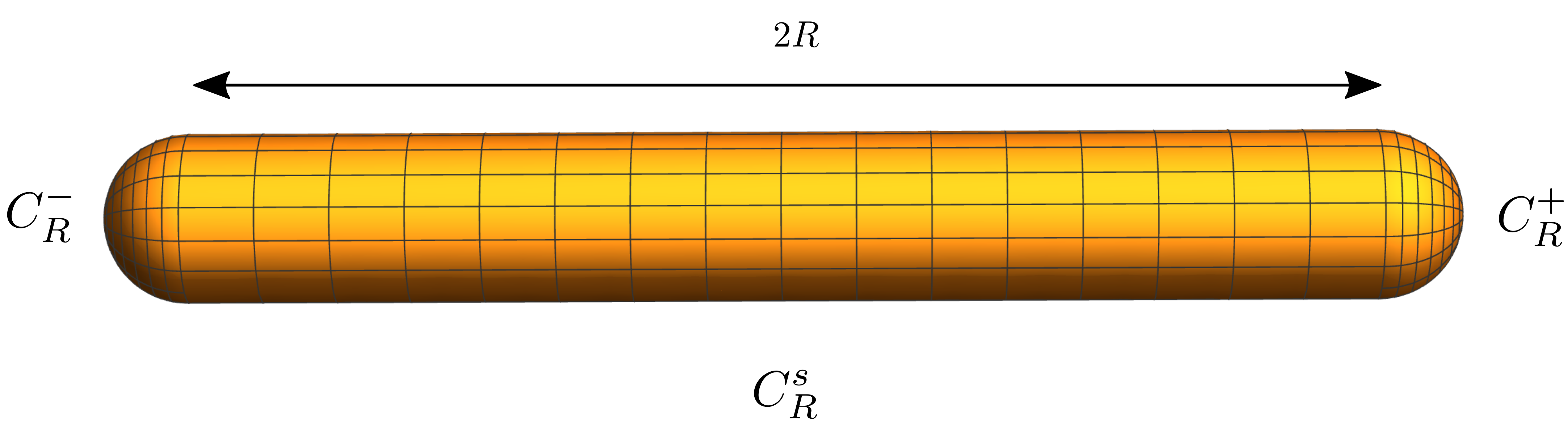
Let D_R be the bounded domain of cylinder shape such that its boundary consists of three parts:

$$\partial D_R = C_R^+ \cup C_R^- \cup C_R^s, \quad C_R^s = \{(x_1, x_2, x_3) : |x'| = 1, -R \leq x_3 \leq R\},$$

and C_R^+ and C_R^- are of arbitrary but fixed shape independent or R. We assume that ∂D_R is $C^{1,\alpha}$-smooth for some $\alpha > 0$.

Theorem 3. *If $\{R_j\}$ be an increasing sequence such that $R_j \to \infty$ as $j \to \infty$, then*

$$\overline{\cup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial\Omega_{R_j}})} \supset [0, 1/2].$$



Outline of Proof of Main Result 3

Let $\mathcal{T}_R = \mathcal{K}_{\partial D_R}$ for simplicity. If ψ is supported in C_R^s, it can be written as $\psi(x) = \psi(\theta, x_3)$, where $x' = (\cos\theta, \sin\theta)$. If $x = (\theta, x_3) \in \Gamma_R^s$,

$$\mathcal{T}_R[\psi](x) = \frac{1}{4\pi}\int_{\mathbb{R}}\int_{-\pi}^{\pi} \frac{1-\cos(\theta-\phi)}{\left[2\left(1-\cos(\theta-\phi)+(x_3-y_3)^2\right)\right]^{3/2}}\psi(\phi, x_3)d\phi dy_3.$$

Taking the Fourier expansion of ψ w.r.t. θ variable $\psi(\theta, x_3) = \sum_{n=-\infty}^{\infty}\psi_n(x_3)e^{in\theta}$,

we have

$$\mathcal{T}_R[\psi](\theta, x_3) = \sum_{n=-\infty}^{\infty} e^{in\theta}\int_{\mathbb{R}}\mathcal{L}_n(x_3-y_3)\psi_n(y_3)dy_3,$$

where

$$\mathcal{L}_n(x_3) = \frac{1}{2\pi}\int_0^{\pi}\frac{(1-\cos\theta)\cos n\theta}{\left[2\left(1-\cos\theta\right)+{x_3}^2\right]^{3/2}}d\theta$$

whose Fourier transformation is

$$\widehat{\mathcal{L}_n}(\xi) = \frac{1}{4\pi}\int_0^{\pi}\cos n\theta\ \widehat{k}(\sqrt{2\left(1-\cos\theta\right)}\xi)d\theta, \quad k(x_3) := \frac{1}{(1+{x_3}^2)^{3/2}}.$$

We have the following properties of $\widehat{\mathcal{L}_n}(\xi)$.

Lemma 3.1 *Let k be defined as above. Then $\widehat{k}(\xi)$ is even, decreasing in $\xi \geq 0$, continuously differentiable, $0 \leq \widehat{k}(\xi) \leq \widehat{k}(0) = 2$, and*

$$\left|\widehat{k}(\xi)\right| \lesssim \frac{1}{1+|\xi|^N} \quad \textit{for any positive integer } N.$$

Outline of Proof. Note that

$$\widehat{k}(\xi) = 2\int_0^\infty \frac{\cos 2\pi\xi t}{\left(1+t^2\right)^{3/2}} dt = 2\pi\xi K_1(2\pi\xi),$$

where K_ν denotes the modified Bessel function of the second kind. Note also that $\left(\xi K_1(\xi)\right)' = -\xi K_0(\xi)$ (See, e.g., "NIST Handbook of Mathematical Functions").

Lemma 3.2 *$\widehat{\mathcal{L}_0}(\xi)$ is even, decreasing in $\xi \geq 0$, continuously differentiable on $\mathbb{R}$, $0 < \widehat{\mathcal{L}_0}(\xi) \leq \widehat{\mathcal{L}_0}(0) = 1/2$, and for any $\delta > 0$*

$$\left|\widehat{\mathcal{L}}_0(\xi)\right| \lesssim \frac{1}{1+|\xi|^{1-\delta}}.$$

We construct test functions as follows. For $\lambda \in (0, 1/2]$, choose $\xi_0 \in \mathbb{R}$ so that

$$\lambda - \widehat{\mathcal{L}_0}(\xi_0) = 0.$$

Put $\rho = R^{1-\sigma}$ for $\sigma \in (0, 1)$. Then, define

$$\psi_\rho = \begin{cases} g_\rho(x_3), & \text{if } x \in C_R^s, \\ 0, & \text{if } x \in C_R^+ \cup C_R^-, \end{cases}$$

where

$$g_\rho(x) = \rho^{-1/2} e^{2\pi i \xi_0 x} (\chi_1 \zeta_1)(\rho^{-1} x).$$

Here, χ_1 is a smooth cut-off function such that $\chi_1 \subset B(0, 1)$ and $\chi = 1$ on $B(0, 1/2)$, and ζ_1 is a smooth cut-off function such that $\int_{\mathbb{R}} \widehat{\zeta_1} d\xi = 1$.

We can prove the following.

Proposition 3.3 *Let $\lambda \in (0, 1/2]$. Then*

$$\lim_{R \to \infty} \frac{\|(\lambda I - \mathcal{T}_R)[\psi_\rho]\|_{L^2(\partial D_R)}}{\|\psi_\rho\|_{L^2(\partial D_R)}} = 0.$$

The rest of the proof is in the same way as before.